Do space-time uncertainty relations require strings?

G. S. Karatheodoris

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1 Introduction

Yoneya [30] has, since at least as early as 1987, vigorously argued that a fundamental, universal (and therefore non-perturbative) aspect of string theory may be an astonishingly simple uncertainty principle involving space-time uncertainties: $\Delta X \Delta T \gtrsim l_s^2$. Given the remarkable breadth of mathematics emerging with and from string theory, and the indispensable role of intuitive physical principles in connecting mathematics with nature, the possibility that such a simple formal expression may maintain its integrity and applicability throughout an amazing morass of mathematics strikes me as a minor miracle. Unfortunately the meaning of the symbols in the above formal expression has not, in general, been worked out; but it has always been difficult to provide an entirely precise statement of what is meant by an uncertainty relation involving time uncertainties–this despite the fact that an experimentally successful one has been available for analysis since the birth of quantum mechanics. In many particular physical situations the meaning of the energy-time uncertainty relation is rather clear. The same expression holds in quantum field theory, but of course there spatial coordinates–demoted to indices–join the time coordinate.

While the role of a space-time uncertainty relation in the formulation of the physical meaning of string theory is being studied, its role as a possibly necessary conclusion following from quantum theory and general relativity has also attracted attention; c.f. the holographic principle. Arguments have been given for the proposition that consistency of the well accepted consequences of quantum field theory and *classical* general relativity may be enough to conclude relations between space and time uncertainties[18]. These arguments have been given in different flavors, e.g. [38] [19], with different conclusions about the microstructure of spacetime. We will call arguments of this type, which attempt to extract new uncertainty relations from local QFT and classical GR, '*operational arguments*', due to the fact that they all rest essentially on gravitational instability toward black hole formation as a mechanism for operationally obstructing any measurements not satisfying the putative uncertainty relations. Conclusions vary; some imply a minimal length in every spatial coordinate and others do not.

1.1 organization and purpose

We wish to show that:

- **Local Qft is unable to accommodate a space-time uncertainty relation.** What is the *physical reason* the space-time uncertainty principle is hard/impossible to implement in the point particle setting? Literature has already been devoted to this subject, some of this has proposed new quantum field theories which manifestly satisfy SURs, but are then argued not to be unitary, other literature has shown that some natural attempts to construct this type of field theory as a decoupling limit of string theory encounters obstacles. (A possible new interpretation of the SUR might be as follows: If one wishes to investigate a feature of spatial size ΔX, one will generically create a BH and will have to wait some time ΔT in order for the information to leak out.)
- **Explain why String theory naturally accommodates the space-time uncertainty principle.** There is a formulation of string theory in which it is *manifestly* satisfied.
- **When a field theory is dual to a string theory we identify the assumption that must be changed in the local QFT framework in order for the SUR to exist.**

One study that may be interesting: examine the scattering of a quantum field off of a classical source $T_{\mu\nu}$. In inverse scattering theory this is called an 'object function'. We should recover the basic result that in order to resolve aspects of the object function (seems to be the classical current in the qft formalism) on a length scale L, we need to scatter waves of wavelength $\lambda \lesssim L$. The object function may be destined to be a black hole at some time in the future all on its own (e.g. it may satisfy the Hoop conjecture), or the system consisting of the probing field and the object may be destined to become a black hole, or the object may already be a black hole. There are two cases here, one in which the object function is classical (as assumed above), the other in which the quantum nature of the object is taken into account.

Suppose we want to resolve a feature of size L in an object of size R, we might want to use *more* energy than is required. Suppose we use so much energy in the probing field that the scattering creates a black hole with radius larger than R, will the information about the feature of size L eventually escape?

The analysis may become quite complicated depending on the 'fine structure' of the object function. If the object function has very fine structure and we wish to resolve it, we have to use modes of the probing field with very high energy, and the deformation of the metric corresponding to this energy density may be very large, perhaps leading to back hole formation. If this black hole formation occurs, the problem of reconstructing the object function from the scattering data becomes entangled with the the problem of black hole information loss. This connexion seems underemphasized in the literature, where it is usually assumed that once a black hole of a certain spatial size L forms, information about spatial structures smaller than L will forever be lost. Since the latter conclusion leads to a manifestly non-unitary evolution as famously recognized by Hawking, the apparent unitarity problems arising from a formalism that essentially assumes black holes swallow this information permanently is hardly surprising.

1.2 On the black hole 'Obstruction'

Arguments that rely on the existence of a black hole 'obstruction' to a process are ubiquitous, especially along the lines of thinking entertained in this paper. They are used frequently to argue against the use of effective field theory techniques when analyzing landscape issues.

It is not however clear that they are correct. In the *classical* theory of gravity, one is *assured* of the creation of a black hole if certain conditions on $T_{\mu\nu}$ are satisfied. For example the Thorne Hoop Conjecture is commonly appealed to. In the standard formulation of quantum field (or string) theory, one conducts experiments with extreme deference to asymptotic data; it is where things are measured and wave-functions collapse–the only place we even *attempt* to make measurements. If one asks detailed questions about the goings on in the interaction region, one is told about a variety of processes that are possible and to what extent they contribute to the total amplitude, but not, of course, which processes 'actually occurred'. Questions like the latter are dealt with precisely as they have been since the two slit experiment, which, as Feynmann famously noted, contains the *only* mystery of quantum theory.

Extending this intuition to a putative 'quantum theory of gravity' which accepts quantum theory wholesale (such as string theory), one would not expect questions about black holes in the interaction region to be dealt with any differently. One would simply examine asymptotic data to make a determination about unitarity or other aspects of the theory.

Hawking has argued that, in the Euclidean path integral approach, amplitude contributions from metrics associated with the Schwarzschild topology are zero, while ones associated with R^4 topology clearly preserve unitarity.

In this clearly quantum gravitational framework one must rephrase the information loss problem. In the two slit experiment, one can destroy the interference pattern by making an observation at a slit to 'determine' whether or not the electron went through it. What is the gravitational analog to this act? If it were possible to block every mode of a process *except* those that required the formation of a black hole, we could recover the information loss problem.

1.3 NCOS

NCOS is the result of introducing a critical electric field and decoupling closed string from open ones. The spacetime (is it still made of closed strings in NCOS theory? If so, the energy density of open strings must not cause spacetime to bend, as open and closed strings are decoupled. It is not clear what is going on here physically.) that the open strings live in is space-time noncommutative. More precisely the endpoints of the open strings do not commute, but the C.M. coordinates do. This is important, since the oscillator modes do not decouple and the result is not a field theory, but a string theory, and thus, while the D-brane world volume exhibits space-time noncommutativity, the entire string is the relevant dynamical object and *it* extends into directions which are commutative (in the absence of a NS magnetic field). Does the $1+1$ dimensional noncritical NCOS theory grow and extra dimension due to a Liouville field? If it does not the $1+1$ theory is a bit strange indeed. In the usual analysis 2d string theory is dual to the $c = 1$ matrix model. Is something like this also true for NCOS in $1+1$? Is seems like it should. If this is true we should have a matrix model description of string theory on a noncommutative spacetime. That could be very interesting in elucidating the role of time in quantum mechanics.

How does Montonen-Olive duality work out?

I would need to:

- Write the NCOS theory in a form that allows the dual matrix model to be written down. Presumably one could take the duality described by Klebanov and Maldacena and further calculate the matrix model that corresponds to the gauge theory. The $U(1)$ factor and the UV/IR effect are essential in understanding the decoupling of the gravitational degrees of freedom. This should be understood in the spirit of Steinacker's work.
- Review NCSYM.
-

2 Why is QFT unable to accommodate a SUR?

2.1 noncommutativity

In standard textbook considerations it may seem that there has never been an attempt or motivation to suggest a space-time uncertainty relation in QFT. In QFT, divergences are dealt with via the ideas of renormalization. These ideas have not only allowed for extremely precise predictions, but have fundamentally shaped the physical interpretation of, and attitude toward QFT that has come to be dominant, namely the *effective field theory* attitude. According to this interpretation, if the QFT is empirically successful at a certain energy scale and is renormalizable, all information about physics at the higher energy scales (e.g. the Planck scale) is exhibited through a finite number of parameters determined numerically through experiment. The only hint we have of physics beyond the scale at which QFT is valid is these numerical values. GUT? They allow the number of parameters to be reduced. The larger the gauge symmetry relating a given number of fields the fewer the arbitrary parameters, but then the game begins again. Also, the natural scale associated with minimum uncertainty in a SUR scenario is the Planck scale, in the absence of gravity, there is no candidate with the status of $\hbar$ for the the right hand side of an SUR.

Long before the conceptual richness and beauty of the renormalization ideas were fully appreciated, Heisenberg conceived the idea of eliminating the divergences of the known QFT's by introducing a "quantum uncertainty" in the location of spacetime points by postulating commutation relations between spacetime coordinates. When Snyder took up this idea he had to assume a fundamental unit of length, a, that had no a priori connexion with gravitation.

2.2 A Holographic reason?

The simple intuitive reason may be that only a *holographic* relativistic theory allows space-time uncertainty relations to hold. The mechanism might be as follows: as one sets up a physical interaction that attempts to determine information in violation of the SUR, the information density in spacetime will violate the holographic principle.

So, how does one attempt to violate the SUR? First of all, how does a field contain information? It does so through its fourier modes. Perhaps it is more convenient to think in terms of microstates and Shannon entropy

$$S = -p_i \log p_i$$

2.3 SURs and minimal length

SURs, being quantum uncertainty relations operate in a more subtle way then lattice type cutoffs, and the phrase 'minimal length' does no justice to the idea. One particular uncertainty may be measured to any accuracy at the expense of a 'conjugate' one.

However in the approach in [40] the standard quantum limit is used to argue for a minimal difference in two consecutive measurements of the eigenvalues of the position operator. They arrive at expressions like $\Delta x > l_{pl}$. This is the kind of expression one expects from a lattice theory, cutoffs in quantum theory are of a different type. While trying to remain flexible, it is important to consider what invariant meaning a minimum length might have. It is rather ironic that this attempt to strictly adhere to the principle of both relativity and quantum mechanics has seemingly lead to a fundamental invariant length agreed upon by all observers; this is manifestly at odds with relativity. We take this as a strong indication that all principles have not been consistently applied.

While the underlying physical reasoning is similar in [18] (both sets of authors use the universal nature of gravitational collapse in conjunction with the basic constraints of quantum mechanics) the proposals are in the end quite different. The proposal in [18] does not have a minimal length. The logic of this approach is to argue heuristically for the 'minimum ball' as in [40] which then implies that every Δx^{μ} can not be zero in the same quantum state (we don't say 'simultaneously zero' as we don't know what that means in this theory). From this point on Doplicher et. al. try to build the theory from the SURs up, using standard consistency conditions from quantum field theory. A set of operators is chosen to realize the SURs, and using the techniques of noncommutative geometry, quantum field theory is written on the noncommutative space generated by these operators. The SURs proposed in [18] are

$$\Delta x^0 \sum_{j=1}^{3} \Delta x_j \gtrsim l_P^2 \tag{1}$$

$$\sum_{j,k=1}^{3} \Delta x_j \Delta x_k \gtrsim l_P^2. \tag{2}$$

It should be noted that the above is not arrived at uniquely in [18], but suggested as a natural and apparently consistent solution to their uncertainty problem. Note the more quantum mechanically conventional form of the above uncertainty relations.

One now has to construct a dynamical theory that lives harmoniously with this kinematical framework. Here is where things get very interesting. In some sense we can see in these approaches expected dynamical results from semiclassical quantum gravity being encoded into a proposed kinematic construction in quantum gravity. The idea is quite old and has proved both successful and unsuccessful. Einstein did it famously with Maxwell's theory yielding special relativistic mechanics as well as special relativistic electrodynamics, but Einstein already had a fully dynamical theory realizing the new kinematics. To be more precise, electrodynamics already had a dynamical symmetry called Lorentz symmetry, Einstein realized that he could 'kinamatize' this symmetry, i.e. identify it directly as a property of space-time and *consequently* a necessary property of not only electrodynamics, but any dynamics. This let to relativistic mechanics. While the insight that dynamical theories should be Lorentz invariant is surely one of the most profound in history, this progress, from an invariant dynamical theory of electrodynamics to an invariant theory of mechanics, was in a disappointing direction. Electrodynamics clearly more deeply connected with nature than particle mechanics c.f. gauge theory. An example of moving in the right direction is given by the history of holography.

3 Unitary and NC

Due to the work of Bahns et. al. [10] noncommutative theories with space-time noncommutativity (i.e. $\theta^{0i} \neq 0$) are unitary and well defined, I responded that there is certainly a vigorous debate going on as to whether the formalism in e.g. [10] was even internally consistent–completely aside from its connection with reality.

Note to myself: [About unitary and noncommutativity with $\theta^{0i} \neq 0$ Doplicher et. al. claimed that their theory is fine. Gomis, Seiberg, Greenberg found lots of problems but never said they could not be overcome by some kind of modification of the theory. Doplicher then changed the theory by using a weird time ordering that had never been used in field theory calculations that were favorably compared with experiment. In other words it was a departure from the usual and verified formalism. SO people started to see if the new time ordering simply pushed the inconsistency (lack of unitarity) elsewhere. Fujikawa found that the positive energy condition (which is essential for spin-statistic theorem) is not satisfied with the new rules and others found violation of the Ward identity. There are big problems with causality studied by greenberg and seiberg. In string theory one finds that in the case of spacetime noncommutativity there is no field theory limit. This means that string oscillator states cannot be ignored, which would explain the non-unitarity problem Doplicher et. al. initially faced. It is of course possible that these problems will be understood in time...]

I am very interested in this issue because of my research too. Some (possibly overlapping) problems discussed in the literature are

- Failure of the Ward identities. Does the holomorphic ward identity imply the Ward identity discussed in [8].
- Failure of Unitarity. Unitarity requires that the
- Instanton type solutions
- Are electric fields required for spacetime noncommutativity? The Yoneya argument implies that the answer is in the negative.
- What happens physically when $\theta^{0i} \neq 0$ that makes the stringy degrees of freedom dynamical? What makes the string wiggle? Also what is the reason for the decoupling of closed modes and not open ones?

4 density of states

Notes taken from [41]. This section is designed to clarify the origin of UV divergences of QFT and the resolution of these in string theory. Three high energy limits:

- High C. M. energy $E = \sqrt{s}$; fixed angle. The Virasoro-Shapiro amplitude for four tachyon scattering is

$$S(k_1, k_2, k_3, k_4) = ie^{2\Phi_0} C_{S_2} (2\pi)^{26} (\sum_i k_i)$$
$$\frac{\Gamma(-\frac{s}{2} - 1)\Gamma(-\frac{t}{2} - 1)\Gamma(-\frac{u}{2} - 1)}{\Gamma(-\frac{s}{2} - \frac{t}{2} - 2)\Gamma(-\frac{t}{2} - \frac{u}{2} - 2)\Gamma(-\frac{u}{2} - \frac{s}{2} - 2)}$$

- Soft scattering: high C.M. energy and small angle, holding fixed the momentum transfer $q \sqrt{-t}$.

5 nonlocal conservation laws

Some work has been done in this area [44].

6 the energy-time uncertainty relation

As is well known, this well verified physical principle stands on a slightly different footing than the position momentum uncertainty relation. This is because time and energy cannot satisfy canonical commutation relations $[H, T] = i\hbar$ (for *any* choice of time operator) without destabilizing the Hamiltonian (i.e. stretching its spectrum to $-\infty$). This is an immediate consequence of the uniqueness aspect of the Stone-von Neumann theorem.

There is however a very simple argument, well known enough to be found in undergraduate textbooks, for why a formally identical "uncertainty relation" should hold despite the dissimilarity in the operator analysis. First of all we must abandon the (nonsensical) notion that Δt represents the standard deviation of a series of measurements of the "time property" of a system. Instead we imagine the system as being characterized by a (possibly over-complete) set of (not explicitly time dependent) observables satisfying the Heisenberg equations of motion, and *regard some appreciable time as having passed for the system* provided that the expectation value of at least one of the observables has evolved by a standard deviation. Consider, for example a set of $\Delta t_{\mathcal{O}}$, each indexed by an operator $\mathcal{O}$. Then

$$\Delta E = \hbar [\min_{\mathcal{O}} \Delta t_{\mathcal{O}}]^{-1}$$

where $\Delta t_{\mathcal{O}}$ is defined by

$$\frac{d}{dt} < \mathcal{O} > \Delta t_{\mathcal{O}} := \sigma_{\mathcal{O}}.$$

(One should include the states $|\psi>$ here. Could this be Wick rotated to an energy-temperature uncertainty relation in field theory? Note also that, since the uncertainty in energy/time is defined in terms of a certain set of observables, which may be appropriate for the system's description in one energy regime, but not in another (c.f. effective field theory point of view) we must understand whether or not the renormalization group preserves the uncertainty relation. Here is a crazy idea: the *physical* time in quantum mechanics is Δt–the time during which *something* happens. The rate of that physical time depends on the energy scale being considered, as faster 'modes' or processes are considered as the range of energy scales being entertained grow. Then the 'end of short distance physics' will then imply...)

First we have to generalize this to QFT and then to string theory. Perhaps the place to start is the scattering formalism in standard quantum mechanics detailed in the Bohm textbook.

6.1 non-relativistic scattering

We wish to examine some examples of valid use of the energy-time uncertainty relation in the Feynmann formalism and then propose a general argument for the energy-time relation within this formalism, attempting not to introduce concepts/techniques from outside. Do the example problems in Griffiths using the path integral!

Standard deviation in the path integral formalism.

The t appearing in the Schrödinger equation is a *parameter* time. The Schrödinger equation is invariant under a simultaneous re-scaling of the time and a reciprocal re-scaling of the Hamiltonian. Thus changing the unit in which time is measured is identical to changing the unit of energy measurement. There is a 'conjugacy' here that can be seen e.g. from

$$E = h\nu$$

For the free non-relativistic particle, observables are e.g. the **position, momentum**, and **energy** of the particle. The position $\hat{X}$, in conjunction with the state, $|\psi(t)>$, and the dynamical equation allow one to calculate the spread in the wave packet as a function of time. This can be used as a "clock" that ticks every time the packet spreads by a standard deviation. The clock function is

$$\sigma^2(t) = \sigma^2(1 + \frac{t^2}{4m^2\sigma^4}) \tag{3}$$

For initially narrow wave packets, $\sigma \ll$ the relation is approximately

$$\sigma(t) = \frac{t}{2m\sigma} \tag{4}$$

7 uncertainty in field theory

7.1 the short time region

The argument in [30] is analyzed. In the 'short time region' (which is not a clearly defined concept), the uncertainty in the energy grows indefinitely. What might replace the formal symbol Δt? In analogy with the non-relativistic analysis, we should consider the most quickly evolving observable characterizing the system, and define the smallest time step as the time required for this observable to evolve appreciably (for example by one standard deviation).

Since there is extreme formal similarity in many particle quantum mechanics and QFT, we should decide at this point which view to adopt. The wonderful conceptual advantages of the field view lead us to adopt it, thus we define Δt in a given physical system S as the time step required to advance the most rapidly changing observable by one standard deviation.

Here we need an example. In the case of a scalar field theory with a source, the observables are the field and its derivative. When has the field undergone a substantial change? Perhaps when a substantial **number of particles** have been created? This seems too vague.

Statistical fluctuation theory and relaxation to equilibrium allow for the calculation of the size of fluctuations and their durations. These can be calculated when the field is isolated in equilibrium or when is has been disturbed by a source. When a field fluctuates, the fluctuations can be of a quantum or thermal nature (or some combination). To what extent are these fluctuations observable? What is the correlation between the size and the lifetime?

How are the uncertainty relations realized in QFT? From the point of view that QM is 0+1 QFT, the same problem that arises with Δt (in QM) also arises with Δx in QFT. The quantum operators are the fields $\hat{\phi}(x)$, and thus the Robertson uncertainty relation is only applicable on field space. The interpretation of Δt, Δx must be considered carefully. One important question is whether Δt is determined by **internal** (quantum) system variables, or by a classical **external** clock (perhaps by taking the classical limit of a quantum environment).

Briggs makes the statement: "A closed quantum system has a time-independent Hamiltonian and there is no reason to introduce time." Is this correct? Is there *no* evolution for a closed QM system? This statement seems to have no basis at all.

The already confusing situation is compounded, at least in magnitude if not in substance, by the replacement of the time parameter by the metrical spacetime continuum, in QFT. Arguing by analogy one might expect that spacetime and the metric might be related to the decohered limit of a quantum environment.

Possibilities for Δt:

- Lifetime of an unstable state
- One standard deviation of an internal clock variable

- One standard deviation of an external 'environmental' clock variable.
- The duration of an experiment (supposedly refuted by Aharonov and Bohm)

How does the metric for a one dimensional (time) line arise from the quantum environment?

Back to the density of states and uv divergences in qft. The basic question is why are there uv divergences in qft and why does string theory avoid them. This is too big a problem to tackle in detail within this article. **We just need an intuitive argument; perhaps even just a sharpening of Yoneya's argument.**

8 the traceless energy momentum tensor

The canonical energy momentum tensor

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \partial_\mu\phi\partial_\nu\phi$$

is

$$\begin{aligned} T^{zz} &= (\bar{\partial}\phi)^2_\star \\ T^{\bar{z}\bar{z}} &= (\partial\phi)^2_\star \\ T^{z\bar{z}} = T^{\bar{z}z} &= m^2 e^{\phi}_\star \end{aligned}$$

for the Liouville theory. If we modify it by introducing an extra term, antisymmetric in the first two indices, that does not spoil the conservation law (due to Belifante) we get,

$$T'^{\mu\nu} = T^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}$$

we can satisfy both the constraints of tracelessness and conservation. We find

$$\begin{aligned} B^{\bar{z}z\bar{z}} &= \epsilon^{\bar{z}z} g^{\bar{z}z}\partial_z \\ B^{\bar{z}zz} &= \epsilon^{\bar{z}z} g^{z\bar{z}}\partial_{\bar{z}} \\ B^{z\bar{z}\bar{z}} &= \epsilon^{z\bar{z}} g^{\bar{z}z}\partial_z . \end{aligned}$$

The resulting traceless, conserved energy momentum tensor is, dropping the prime,

$$T^{\bar{z}z} = T^{z\bar{z}} = m^2 e^{\phi}_\star + 2\partial\bar{\partial}\phi = 0 \tag{5}$$

$$T^{zz} = (\bar{\partial}\phi)^2_\star, \quad T^{\bar{z}\bar{z}} = (\partial\phi)^2_\star \tag{6}$$

As we will show conservation equation simply states, up to Moyal brackets. In the commutative case the equations read,

$$\tilde{T}(\bar{z}) = T^{zz}(\bar{z}) = 4T_{\bar{z}\bar{z}}(\bar{z}) \tag{7}$$

and

$$T(z) = T^{\bar{z}\bar{z}}(z) = 4T_{zz}(z) \tag{8}$$

The split is into anti-holomorphic and holomorphic parts respectively. In the noncommutative case the matter is a bit more complicated, the conservation equations are,

$$\partial T^{zz} = -\frac{1}{2}m^2\left(e_\star^\phi \star \bar\partial\phi + \bar\partial\phi \star e_\star^\phi + \bar\partial e_\star^\phi\right) \sim 0$$
$$\bar\partial T^{\bar z\bar z} = -\frac{1}{2}m^2\left(e_\star^\phi \star \partial\phi + \partial\phi \star e_\star^\phi + \partial e_\star^\phi\right) \sim 0$$

The $\sim$ sign means up to Moyal brackets, it is used because arbitrary Moyal brackets may appear on the right hand side without affecting the properties hitherto discussed. To pursue the matter further we have to evaluate the derivative of a star exponential.

$$\partial e_\star^\phi = \partial \sum_{n=0}^{\infty} \frac{1}{n!}\phi_\star^n$$

Now, the derivative of each term $\phi_\star^n$ is

$$\partial\phi_\star^n = \partial\phi \star \phi_\star^{n-1} + \phi \star \partial\phi \star \phi_\star^{(n-2)} + \cdots + \phi_\star^{(k-1)} \star \partial\phi \star \phi_\star^{(n-k)}$$

Note there are n terms in $\partial\phi_\star^n$. Now we can sum each of these terms (labeled with a k) over n. For the first we get

$$\sum_{n=0}^{\infty} \frac{1}{n!}\partial\phi \star \phi_\star^{n-1} = \partial\phi \star \phi^{-1} \star e_\star^\phi$$

The second is

$$\sum_{n=0}^{\infty} \frac{1}{n!}\phi \star \partial\phi \star \phi_\star^{n-2} = \sum_{n=0}^{\infty} \frac{1}{n!}(\partial\phi \star \phi + [\phi, \partial\phi]_M) \star \phi_\star^{n-2} =$$

(the M stands for Moyal bracket in order to eliminate at least some stars from the equations)

$$\partial\phi \star \phi^{-1} \star e_\star^\phi + [\phi, \partial\phi]_M \star \sum_{n=0}^{\infty} \frac{1}{n!}\phi_\star^{n-2} = \partial\phi \star \phi^{-1} \star e_\star^\phi + [\phi, \partial\phi]_M \star e_\star^\phi \phi_\star^{-2}$$

using induction we prove, for the $\mathbf{k}^{\text{th}}$ term

$$\left(\partial\phi \star \phi^{-1} + [\phi, \partial\phi]_M \star \phi_\star^{-2} + \cdots + [\phi_\star^{k-1}, \partial\phi]_M \star \phi_\star^{-k}\right) e_\star^\phi$$

or

$$\sum_{l=0}$$

8.1 some star products

In order to get a feel for the new product we review its action on holomorphic and anti-holomorphic functions. First note

$$[z,\bar{z}]_\star = [z,\bar{z}]_M = 2\theta^{z\bar{z}} = 2\theta\epsilon^{z\bar{z}} \tag{9}$$

where ϵ is defined as usual. For these formulas are easily deduced from the differential expression of $\star$, namely,

$$\star = e^{\theta^{\mu\nu}\partial_\mu\partial_\nu}$$

From straightforward computations we have

$$\begin{aligned}
z^n \star z^m &= z^{m+n} \\
\bar{z}^n \star \bar{z}^m &= z^{m+n} \\
z \star \bar{z} &= |z|^2 + \theta \\
\bar{z} \star z &= |z|^2 - \theta \\
z^m \star \bar{z} &= |z|^2 z^{m-1} \\
f(z) \star \bar{z} &= \bar{z} f(z) + \theta \frac{\partial f}{\partial z} \\
\bar{z} \star f(z) &= \bar{z} f(z) - \theta \frac{\partial f}{\partial z}
\end{aligned}$$

$$[f(z),\bar{z}] = 2\theta \frac{\partial f}{\partial z}$$

$$\begin{aligned}
f(z) \star \bar{z}^2 &= \bar{z}^2 f(z) + \theta \frac{\partial f}{\partial z}(2\bar{z}) + \frac{1}{2}\theta^2 \frac{\partial^2 f}{\partial z^2} \\
f(z) \star \bar{z}^m &= (\theta\partial + \bar{z})^m f(z)
\end{aligned}$$

The most important formula comes from the last identity generalizing to an anti-holomorphic function $g(\bar{z})$.

$$f(z) \star g(\bar{z}) = g(\theta^{\bar{z}z}\partial_z + \bar{z}) f(z) \tag{10}$$

On the right hand side g is no longer a function on $\mathbb{C}$, but rather an operator on sufficiently differentiable functions $f : \mathbb{C} \to \mathbb{C}$.

(For the uninitiated: There is a small abuse of notation here; $f(z) \star g(\bar{z})$ does not mean anything. We use it here as an alternative to a more honest notation $(f \star g)(z,\bar{z})$, where the reader would have to remember the analaticity properties of the factors. This may seem like a trivial point, but the confusion appears in the literature. Note $f(3+4i) \star g(5+7i)$ is not defined, but $(f \star g)(3+4i)$ is well defined and depends on the values of f and g on the whole domain, not just at $(z = 3+4i)$.

8.2 Lie product

It turns out notation for a Lie product is convenient for computations in this theory. The Lie product of ϕ with a (not star-commuting) function Q is written

$$(\mathbf{Ad}_\phi)^n Q := [\phi[\phi[\cdots[\phi, Q]\cdots]]]_M, \quad (n - times).$$

Where the subscript M means all brackets are Moyal star brackets. We want to satisfy

$$\frac{1}{2}(\partial\phi \star e^\phi_\star + e^\phi_\star \star \partial\phi) - \partial e^\phi_\star \sim 0 \tag{11}$$

$$\frac{1}{2}(\bar\partial\phi \star e^\phi_\star + e^\phi_\star \star \bar\partial\phi) - \bar\partial e^\phi_\star \sim 0 \tag{12}$$

where $\sim$ means "up to a total commutator". Concentrating first on $\partial e^\phi_\star$ we have

8.3 adjoint identities

Here are a few identities.

$$\mathbf{Ad}_\phi(AB) = A\mathbf{Ad}_\phi(B) + \mathbf{Ad}(A)B \tag{13}$$

That is, $\mathbf{Ad}$ is a derivation. Also $\mathbf{Ad}_\phi$ commutes with any function $B(\phi)$ when acting on Q:

$$[\mathbf{Ad}_\phi, B(\phi)]Q = 0 \tag{14}$$

Furthermore one can deduce the reduction formula,

$$\mathbf{Ad}_{\phi^m} = \mathbf{Ad}_{\phi^{m-1}}(\phi - \mathbf{Ad}_\phi) + \phi^{m-1}\mathbf{Ad}_\phi. \tag{15}$$

This identity then implies that $\mathbf{Ad}_{\phi^m}$ can be written as an operator $\mathcal{F}(\phi; \mathbf{Ad}_\phi)$. To find the form of $\mathcal{F}(\phi; \mathbf{Ad}_\phi)$ we use standard commutator rules to expand

$$\begin{aligned}\mathbf{Ad}_{\phi^m} &= (-1)^{(m+1)} \sum_{k=0}^{m-1} \binom{m}{k} (-\phi)^k \mathbf{Ad}_\phi^{(m-k)} \\ &= (-1)^{m+1}((\mathbf{Ad}_\phi - \phi)^m + \phi^m) \\ &= -(-\phi)^m - (\phi - \mathbf{Ad}_\phi)^m\end{aligned}$$

Where $(\mathbf{Ad}_\phi)^0 := 1$.

The quantity of interest is

$$\partial e^\phi_\star = \sum_{n=0}^{\infty} \frac{1}{n!}\partial\phi^n_\star = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{n} \phi^{k-1}_\star \star \partial\phi \star \phi^{n-k}_\star$$

while

$$\phi^{k-1}_\star \star \partial\phi \star \phi^{n-k}_\star = \frac{1}{2}\partial\phi \star \phi^{n-1}_\star + \frac{1}{2}\phi^{n-1}_\star \star \partial\phi$$
$$-\frac{1}{2}[\partial\phi, \phi^{k-1}_\star]_M \star \phi^{n-k}_\star - \frac{1}{2}\phi^{k-1}_\star \star [\phi^{n-k}_\star, \partial\phi]_M$$

Using the above identities,

$$\frac{1}{2}\partial\phi \star \phi_\star^{n-1} + \frac{1}{2}\phi_\star^{n-1} \star \partial\phi - \frac{1}{2}(\partial\phi)\overleftarrow{\mathbf{Ad}}_{\phi^{k-1}} \star \phi_\star^{n-k} - \frac{1}{2}\phi_\star^{k-1} \star \overrightarrow{\mathbf{Ad}}_{\phi^{n-k}}(\partial\phi) =$$

$$\frac{1}{2}\partial\phi \star \phi_\star^{n-1} + \frac{1}{2}\phi_\star^{n-1} \star \partial\phi - \frac{1}{2}(\partial\phi)(-1)^{k-1}(\overleftarrow{\mathbf{Ad}}_\phi - \phi)^{k-1} \star \phi_\star^{n-k} - \frac{1}{2}(-1)^{n-k}\phi_\star^{k-1}(\overrightarrow{\mathbf{Ad}}_\phi - \phi)^{n-k}(\partial\phi)$$

Making the summation over k yields

$$\begin{aligned}
&\frac{n(n-1)}{2}\left(\frac{1}{2}\right)(\partial\phi \star \phi_\star^{n-1} + \phi_\star^{n-1} \star \partial\phi) \\
&-\frac{1}{2}(\partial\phi)(\phi - \overleftarrow{\mathbf{Ad}}_\phi)^{-1}\sum_{k=1}^{n}(1 - \overleftarrow{\mathbf{Ad}}_\phi\phi^{-1})^k \star \phi_\star^n \\
&+\frac{1}{2}(\phi - \overrightarrow{\mathbf{Ad}}_\phi)^n\phi^{-1}\sum_{k=1}^{n}(1 - \phi^{-1}\overrightarrow{\mathbf{Ad}}_\phi)^k(\partial\phi) \\
&= \frac{n(n-1)}{2}\left(\frac{1}{2}\right)(\partial\phi \star \phi_\star^{n-1} + \phi_\star^{n-1} \star \partial\phi)
\end{aligned}$$

Another goal is to write $\partial e_\star^\phi$ in a way that makes its properties transparent and proves that the conservation law is satisfied (in the noncommutative case this means that the divergence is zero up to a Moyal bracket). We have to cope with expressions like $\phi_\star^{k-1} \star \partial\phi \star \phi_\star^{n-k}$

$$(\phi^{-1}\overrightarrow{\mathbf{Ad}}_\phi)Q = Q(\overleftarrow{\mathbf{Ad}}_{\phi^{-1}}\phi) \tag{16}$$

8.4 conservation to first order in θ

Expanding the conservation equation to first order requires expanding $e_\star^\phi$. First we calculate $\phi_\star^n$.

8.5 Traceless and divergenceless

Tracelessness is only important insofar as it leads to the (anti-)holomorphic split. This split comes from the divergence equation. In the noncommutative case, we find that the divergence

8.6 charges in space-time noncommutativity

In the excellent article [39] it is suggested that the the concept of conserved charge encounters difficulties when θ_{0i} is nonzero. This would present somewhat of a puzzle since space-time noncommutativity in string theories (like little string theories) seems to be constructible without conceptual difficulty and one would expect the conservation of charge not to depend on the higher harmonics of the string (which is what takes care of the unitarity problems that plague space-time noncommutative theories with no extended degrees of freedom. Of course α' effects can fix a lot. One could however imagine that without a clear Hamiltonian formulation charge conservation may be ill-defined. The situation is still obscure to the author. These worries have led to an attempt to write a conserved charge for $\theta_{0i} \neq 0$. The solution seems to be immediate. First note that

$$\int [f,g]h(x)d^{2n}x = 0$$

independently of $h(x)$. Now suppose h is a delta function picking out a spacial slice along which one will define a conserved charge from the noncommutative Noether current defined in [39]. The make things simple, we work in $d = 4$ and take the hyper-surface in the d3elta function to be defined be $t = 0$; we actually have to take two such surfaces to reproduce the standard 'tube lemma' of classical field theory. Then

$$\int [f,g]\delta(t)d^4x = \int [f,g]_M d^3x$$

This means that when we integrate a current that is conserved up to noncommutative brackets, as occurs in general noncommutative field theory,

$$\partial_\mu J^\mu = [f,g]_M$$

we find that

$$\partial_0 Q := \partial_0 \int J^0 d^3x + \int \vec{\nabla} \cdot \vec{J} d^3x - \int [f,g]_M d^3x = 0$$

as expected. In the case of Liouville theory the 'corrections' to the conservation law (we use quotes because the conservation law still provides a number that does not change as time passes, but the equation expressing this fact has changed form) will contain higher order terms in θ.

9 conformal invariance

It is a standard result that if a Poincare' invariant theory possesses scale invariance, and has an energy-momentum tensor that can be made traceless, the theory is fully conformally invariant. The stress tensor for Liouville theory may be made traceless, with an appropriate conformal modification which does not spoil the continuity equation. The connection between tracelessness and conformal invariance is included (in any dimension) for completeness. First let $x^\mu \mapsto x^\mu + \epsilon^\mu$ be an arbitrary coordinate transformation. Then,

$$\delta S = \int d^dx T^{\mu\nu}\partial_\mu \epsilon_\nu$$
$$= \frac{1}{2}\int d^dx T^{\mu\nu}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu),$$

but now, a conformal transformation is defined by

$$\partial_{(\mu}\epsilon_{\nu)} = \frac{2}{d}\partial_\rho\epsilon^\rho$$

thus

$$\delta S = \frac{1}{d}\int d^dx T^\mu_\mu \partial_\rho\epsilon^\rho = 0$$

Since the Liouville field is most conceptually at home the expression $g = e^{\gamma\phi(z,\bar{z})}\hat{g}$, this is consistent with the well known fact that the theory with a set of scalars coupling to gravity in two dimension is always conformal. In this context the Liouville equation appears as $T^\mu_\mu = 0$ A consequence of this is that the nonzero components of the energy momentum tensor split into holomorphic and antiholomorphic pieces.

10 the Virasoro charges

The operator coefficients of the Laurent expansions,

$$T_{zz}(z) = \sum_{m=-\infty}^{m=\infty}\frac{L_m}{z^{m+2}}, \quad T_{\bar{z}\bar{z}}(\bar{z})\sum_{m=-\infty}^{m=\infty}\frac{L_m}{\bar{z}^{m+2}}$$

of the holomorphic and antiholomorphic parts of the energy momentum tensor satisfy the Virasoro algebra if the theory is conformal. The coefficients are give by,

$$L_m = \oint_C \frac{dz}{2\pi i z} z^{m+2} T_{zz}(z)$$

with C counterclockwise.

11 Operator Product Expansions

Here we want to compute the OPE's of the energy-momentum tensor of both the commutative and noncommutative Liouville theories.

12 Ward Identities

Noether's theorems relate local or global symmetries of a Lagrangian to conserved currents or charges respectively. The Noether theorems have a fundamental place in classical Lagrangian and Hamiltonian theories, whether gravitational or not. What does this have to do with the Ward Identities? They are the Quantum Version of the Noether Identities. Using the Ward identities one can (and I did this on the chalk board in the string theory talks) work "backward" from a conserved current to calculate the infinitesimal transformation under which a theory is invariant. Work in 1+1 and analyze Ward.

I tell you this so you can answer the question, "Who cares if the Ward identities are violated?"

First consider a symmetry of the quantum theory with classical action $S[\phi]$.

$$\phi_\alpha(\sigma) \to \phi'(\sigma)_\alpha = \phi_\alpha(\sigma) + \delta\phi_\alpha(\sigma)$$
$$[d\phi]e^{-S[\phi]} \to [d\phi']e^{-S[\phi']} = [d\phi]e^{-S[\phi]}$$

We cow consider a change of variables that is not a symmetry but becomes the above when $\rho \to$ const.

$$\phi_\alpha(\sigma) \to \phi'(\sigma)_\alpha = \phi_\alpha(\sigma) + \rho(\sigma)\delta\phi_\alpha(\sigma)$$

Where $\rho(\sigma)$ is an arbitrary function with compact support. Clearly the change in the path integral must be proportional to $\partial_\alpha\rho$, as the change is zero when ρ is constant.

$$[d\phi']e^{-S[\phi']} = [d\phi]e^{-S[\phi]}(1 + \frac{i\epsilon}{2\pi}\int d^d\sigma\sqrt{g}j^a(\sigma)\partial_a\rho(\sigma) + O(\epsilon^2))$$

To get the Noether theorem we can take a path integral with arbitrary insertions outside the support of ρ, recovering the fact that $\nabla_a j^a = 0$. To get the Ward identity, we take ρ to be equal one on its domain of support, and include one operator $\mathcal{A}(\sigma_0)$ in the support Ω and arbitrary insertions elsewhere. Computing the change in the path integral gives

$$\int_{\partial R} dAn_a j^a\mathcal{A}(\sigma_0) = \frac{2\pi}{i\epsilon}\delta\mathcal{A}(\sigma_0).$$

Here n_a is the outward normal to ∂R. In other words the integral of a conserved current around an operator gives the variation of the operator under the symmetry corresponding to that current.

The free scalar field is invariant under the Poincare group. The Noether charge associated with spacetime translations is the stress-energy tensor.

$$T_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}\partial_c\phi\partial^c\phi$$

Specializing to two dimensions we work in complex coordinates. It is unclear if one can do computations in Euclidean space and continue back to Minkowski space in the noncommutative spacetime (which might be another project). For now we work directly in complex space.

$$z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2$$

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$$

$$(g_{\mu\nu}) = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

Define a current associated with the energy-momentum tensor as follows

$$j_\mu = iv^\nu T_{\mu\nu}. \tag{17}$$

The energy-momentum tensor appearing here is the modified traceless one.

$$T^{zz} =$$

13 Liouville theory

14 noncommutatve Liouville theory

Consider Liouville theory in 2d, with ordinary products replaced by deformed ones. (Double check factors of 1/2 in measure.)

$$S[\phi] = \int_R d^2z \Big(\frac{1}{2} \partial\phi \star \bar{\partial}\phi - \frac{1}{2} m^2 e_\star^\phi \Big) \tag{18}$$

We initially assume that $\partial R = \emptyset$ so quadratic terms may be evaluated pointwise. This new theory is probably not conformally invariant, due to the length scale introduced by θ. Is this theory conformally invariant?

The equation of motion is

$$\bar{\partial}\partial\phi + \frac{1}{2} m^2 e_\star^\phi = 0 \tag{19}$$

The energy-momentum tensor is

$$T^{\mu\nu} = -g^{\mu\nu}\mathcal{L} + \partial^\mu\phi \star \partial^\nu\phi \tag{20}$$

$$\begin{aligned} T^{zz} &= (\partial\phi)^2 \\ T^{\bar{z}\bar{z}} &= (\bar{\partial}\phi)^2 \\ T^{\bar{z}z} = T^{z\bar{z}} &= m^2 e_\star^\phi \end{aligned} \tag{21}$$

The conservation equations read

$$\bar{\partial}(\partial\phi)_\star^2 + \frac{1}{4} m^2 \partial e_\star^\phi = 0 \tag{22}$$

$$\partial(\bar{\partial}\phi)_\star^2 + \frac{1}{4} m^2 \bar{\partial} e_\star^\phi = 0 \tag{23}$$

Using,

$$\partial e_\star^\phi = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{1}{n!} \phi_\star^k \star \partial\phi \star \phi_\star^{(n-1)} \star \phi_\star^{-k} \tag{24}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\frac{1}{n!}\phi_\star^k \star \partial\phi \star \phi_\star^{-k} \star \phi_\star^{(n-1)} \tag{25}$$

With a little fiddling, one can write

$$[\sum_{k=0}^{n-1}\phi_\star^k \star \partial\phi \star \phi_\star^{-k}, \phi]_\star = (\partial\phi - \phi_\star^n \star \partial\phi \star \phi_\star^{-n}) \star \phi \tag{26}$$

Now lets evaluate

$$\partial e_\star^\phi = \partial\sum_{n=0}^{\infty}\frac{1}{n!}\phi_\star^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\frac{1}{n!}\phi_\star^k \star \partial\phi \star \phi_\star^{-k} \star \phi_\star^{n-1}. \tag{27}$$

Let's concern ourselves with

$$\sum_{k=0}^{n-1}\phi_\star^k \star \partial\phi \star \phi_\star^{-k}$$

first. We can write this in terms of nested commutators. In order to make the notation readable,star products will be dropped and all products should be understood as noncommutative (stars) unless otherwise noted. The general form is

$$\sum_{k=0}^{n-1} D_{k,n}\mathbf{Ad}_\phi^k(\partial\phi)\phi^{-k} = \sum_{k=0}^{n-1} E_{k,n}\Big\{\mathbf{Ad}_\phi^k(\partial\phi\phi^{-k}) + \mathbf{Ad}_{\phi^{-1}}^k(\phi^k\partial\phi)\Big\}$$

where $D_{k,n}$ and $E_{k,n}$ are combinatorial factors, $\mathbf{Ad}_\phi(\partial\phi) = [\phi, \partial\phi]_\star$, and higher powers of the latter are nested commutators with correspondingly higher powers of θ. We can also use

$$\overrightarrow{\mathbf{Ad}}_\phi^n = (-1)^n\overleftarrow{\mathbf{Ad}}_\phi^n$$

and the similar relation for $\mathbf{Ad}_{\phi^{-1}}$ in order to simplify things. Now it becomes

$$\sum_{k=0}^{n-1} D_{k,n}\mathbf{Ad}_\phi^k(\partial\phi)\phi^{-k} = \sum_{k=0}^{n-1} E_{k,n}\Big\{(-1)^n(\partial\phi)(\overleftarrow{\mathbf{Ad}}_\phi\phi^{-1})^k + (\phi\overrightarrow{\mathbf{Ad}}_{\phi^{-1}})^k(\partial\phi)\Big\} \tag{28}$$

To achieve some clarity regarding the form of the theory we have to calculate the coefficients $E_{k,n}$ and recognize the resulting function after summation. We can do this using recursion relations.

First we start out with a basic formula and work out a couple of cases explicitly

$$\phi Q\phi^{-1} = 2Q - Q(\overleftarrow{\mathbf{Ad}}_\phi\phi^{-1}) - (\phi\overrightarrow{\mathbf{Ad}}_{\phi^{-1}})Q$$

where Q is any operator, in our case $\partial\phi$. Iterating again gives

$$\phi^2 Q\phi^{-2} = 4Q - 3Q(\overleftarrow{\mathbf{Ad}}_\phi\phi^{-1}) - 3(\phi\overrightarrow{\mathbf{Ad}}_{\phi^{-1}})Q + 2Q(\overleftarrow{\mathbf{Ad}}_\phi\phi^{-1})^2 + 2(\phi\overrightarrow{\mathbf{Ad}}_{\phi^{-1}})^2 Q.$$

In my green notebook, page 74, I came to the conclusion that

$$\partial e_\star^\phi = [A(1-A)]^{-1} e^{(1-A)\phi} \partial\phi \tag{29}$$

where A is an operator defined as

$$A = \phi \mathbf{Ad}_{\phi^{-1}}.$$

Assuming that this is accurate, although it must be checked, we proceed in calculating the divergence of the stress tensor. All the

$$\bar{\partial}(\partial\phi)_\star^2 + m^2 \partial e_\star^\phi = (\partial\phi)\bar{\partial}\partial\phi + \bar{\partial}\partial\phi(\partial\phi) + m^2 [A(1-A)]^{-1} e^{(1-A)\phi} \partial\phi \tag{30}$$

A simplified derivation of the derivative of $\partial e_\star^\phi$ starts with

$$\partial e_\star^\phi = \sum_{n=0}^{\infty} \frac{1}{n!} \phi_\star^n$$

noting

$$\partial \phi_\star^n = \sum_{k=0}^{n-1} \phi_\star^k \star \partial\phi \star \phi_\star^{-k} \star \phi_\star^{n-1}$$

The factor $\phi_\star^k \star \partial\phi \star \phi_\star^{-k}$, can be written as $(1-A)^k \partial\phi$ where,

$$A := \phi \mathbf{Ad}_{\phi^{-1}}$$

Then

$$\begin{aligned}
\partial e_\star^\phi &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n-1} (1-A)^k \partial\phi \star \phi_\star^{n-1} \\
&= -\sum_{n=0}^{\infty} \frac{1}{n!} \frac{(1-A)^n - 1}{A} \partial\phi \star \phi_\star^{n-1} \\
&= -A^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} [(1-A)^n - 1] \partial\phi \star \phi_\star^{n-1} \\
&= -A^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} (1-A)^n \partial\phi \star \phi_\star^{n-1} + A^{-1} \sum_{n=0}^{\infty} \frac{1}{n!} \partial\phi \star \phi_\star^{n-1}
\end{aligned}$$

Note that the operator $(1-A)$ bumps $\partial\phi$ over one space: $(1-A)\partial\phi \star \phi_\star^{n-1} = \phi \star \partial\phi \star \phi_\star^{n-2}$. To sum this series we note that if we define the operator B as an operator acting on the right i.e. $B := \overleftarrow{\mathbf{Ad}}_\phi \phi^{-1}$, we find the relation $AQ = QB$, where Q is an arbitrary $\star$-function of ϕ and its derivatives. With this we can write

$$(1-A)^n Q = Q(1-B)^n$$

The final result is

$$-A^{-1}(\partial\phi) \star e_\star^{\phi(1-B)} (1-B)^{-1} - A^{-1}(\partial\phi) \star e_\star^\phi \star \phi \tag{31}$$

The classical limit must be checked carefully.

Another identity,

$$[\phi \mathbf{Ad}_{\phi^{-1}}, \phi^{-1}\mathbf{Ad}_{\phi}]_{+} = 0$$

What is the inverse of $\phi \mathbf{Ad}_{\phi^{-1}}$? Let's consider first the more familiar problem from quantum mechanics.

$$[x, p]_\star = i\hbar$$

whence,

$$[f(x), p]_\star = i\hbar \frac{\partial f}{\partial x}$$

So to solve for f we must integrate. This is to be expected, as derivations are normally defined in terms of commutators in noncommutative geometry. The operator analog of integration is the trace operation, so we might expect that the inverse of $\phi \mathbf{Ad}_{\phi^{-1}}$ to be given by some kind of trace or integral.

To implement this idea we may need some kind of Hamiltonian structure in order to identify the canonical momentum to ϕ. This chain of ideas may signify trouble due to the lack of a transparent Hamiltonian formulation of noncommutative resp. nonlocal theories.

15 noncommutative spacetime

In the $\{x^0, x^1\}$ coordinates

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} = i\epsilon^{\mu\nu}\theta$$

and therefore in the lightcone coordinates

$$[x^+, x^-] = 2i\theta$$

This algebra can be realized via the star product

$$(f \star g)(x^+, x^-) = \lim_{x' \to x} e^{2i\theta(\partial'_+\partial_- - \partial'_-\partial_+)} f(x'^+, x'^-) g(x^+, x^-).$$

We now use the definition of noncommutative field theory suggested in [13].

$$S[\phi] = \int d^2x \Big\{ \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4_\star \Big\}$$

is the action. In [13] an unorthodox prescription for time ordering is provided which makes the perturbative S-matrix formally unitary. We say formally because we take the view that, at present, the internal consistency of spacetime noncommutative *field* theory is not at all clear. Consistency checks on the suggestion in [13] has given disappointing results and it would be nice to understand why in an intuitive physical way. Arguably such an intuitive reason has already been given in [7], which amounts to the insight that, because there is no field theory decoupling limit in string theory that would leave the spacetime noncommutativity, one must include stringy degrees of freedom in order to preserve unitarity. In other words the string theory analysis provides the missing degrees of freedom. A question is, 'How does this general analysis square with the results from NCOS and OM theories?" Presumably, either further modifications or reinterpretation of the field theory formalism are required or the proposal in [13] The classical equation of motion is

$$(\Box + m^2)\phi(x) = -\frac{\lambda}{3!}\phi_\star^3(x).$$

This result does not depend on the structure of $\theta^{\mu\nu}$. Under a Lorentz transformation, the space and time parts of $\theta^{\mu\nu}$ mix. The space and time parts correspond to magnetic and electric parts of the NS B-field resp.

$$\theta_t^i := \theta^{0i}$$

$$\theta_s^i := \frac{1}{2}\epsilon^i_{jk}\theta^{jk}$$

I am confused about when the unitarity problems arise. Is it just when $\theta_t^i \neq 0$ or when $\theta^{\mu\nu}$ is not space or light like. The standard results [1] indicate that $\theta_t^i = 0$ is necessary for unitarity. However there are duality arguments that seem to contradict this. Aschieri [20] has found a correspondence between static solutions in space-NCDBI and static solution in spacetime-NCDBI. NCOS [24] [21]. OM theory [23] [24] [24].

The shed some light on this we review the S-duality of the DBI theory given by

$$S_{DBI}[A_i; g_{ij}, B_{ij}] = \frac{1}{g_s}\int d^{10}\sqrt{\det(g + 2\pi\alpha'(B+F))}$$

16 the new rules in discrete time theories

17 NC Oscillators

Can one write a model of coupled noncommutative harmonic oscillators forming a 'mattress', (as in Zee) that produces noncommutative QFT?

18 Peierls bracket

Suppose a system's action is perturbed by the addition of a 'small' physical observable ϵB.

$$S[\phi^i] \to S[\phi^i] + \epsilon B[\phi^i]$$

The Peierls bracket is defined as

$$(A, B) := D_A^- B - D_B^- A = D_A^- B - D_A^+ B =$$

19 instability and rigged Hilbert space

exponential versus power law non-normalizability of wave functions makes a difference.

20 the beautiful physical diplole picture

Dipoles moving orthogonally to their extension. The canonical structure is not

21 Schwinger action principle

22 BJL method

23 Yang-Feldman quantization

24 unitarity

The physical basis of unitarity is the inclusion of all physically relevant degrees of freedom. A unitary theory is self contained in the sense that all states into which a given state may dynamically evolve are present in the Hilbert space. If a system contains a subsystem, e.g. a particle that is unstable, the state representing this subsystem must not appear in the asymptotic Hilbert space, since, due to its instability, it cannot freely travel arbitrarily far away from the interaction region.

A simple model due to Veltman, with these characteristics is

Discrete time theories do not satisfy this: " Thus, the theories with a discrete time variable do not satisfy the unitarity condition." Of course there are noncommutative theories with continuous time. For the discrete time case there are un-physical non-oscillating modes.

25 asymptotic completeness

26 Slavanov-Taylor identity

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