– Add a section on exactly how string theory deals with these difficulties.

Generalized gauge theory

March 20, 2012

1 Introduction

Here we develop the notion of a generalized gauge theory. It expands Weyl's notion of local gauge invariance to include a general situation in which symmetry (as defined by structure constants associated with a Lie algebra) ~~is only locally~~ valid as an approximation in ~~a~~ the neighborhood of ~~a~~ point in space-time. The mathematical basis for this generalization has been explored in the mathematics literature under the rubric groupoids.

nonsingular

Why don't we have a useable quantum theory of gravity? The problem of assigning, in a rational way, an energy density to a region of space-time that includes energy of the gravitational field itself has been a longstanding one. It is a bizarre and unexpected problem to beset what has often been described as the ideal classical theory; a theory with only one flaw: lack of a quantum counterpart. In this article we will argue that the problems just cited are in fact interrelated.

the

decisive flaw

The inability to consistently and clearly define the energy density of the gravitational field in all possible physical circumstances to which general relativity applies, has been a blemish on what is ~~thought of as the perfect classical theory.~~ otherwise a ~~perfect~~ classical theory. The fault of general relativity is ~~thought to be~~ its inability to be reconciled with conventional quantum theory. The thesis of this note is that these two problems are related. In fact it seems clear that, within the current framework of quantum theory, any classical theory that forbids the construction of a Hamiltonian will be manifestly un-quantizable. This is the reason that supersymmetry is so very important within the realm of string theory techniques; when some supersymmetry is present a Hamiltonian can be constructed. Thus, from this point of view, string theory is a unified formalism for writing down supersymmetric theories of quantum gravity, within conventional quantum mechanics. This view also leads one to guess that critiques of the landscape scenario are correct. All of this can be seen as a simple way of understanding where the difficulties in progress lie, without refercing to renormalization arguments. ~~We think that,~~ While renormalization arguments must be addressed, they are (may be) a ~~bad~~ poor starting point for finding correct directions in fundamental theory.

It is widely agreed that

superb

should be rewritten.

→ They lead naturally to multifarious, creative formal regularized schemes and with fingers crossed one hopes principles will emerge.

In order to solve the quantum gravity problem, we must decide between two equally reasonable sounding alternatives.

This may allow for Hilbert spaces with changing dimension

Groupoid Quantum Mechanics.

- Adjust quantum theory in order that it admit theories that do not have a traditional Hamiltonian.

- Fi~~x general relativity such that it is equipped with a traditional Hamiltonian.~~ Replace GR by a scheme in which a Hamiltonian can be identified.

2 Algebroids

Prove equivalence to Tresguerres' composite bundle

We ~~here~~ review a construction within the theory of Lie algebroids ~~which is the infinitesimal version of the Lie groupoid~~. A Lie algebroid is a vector bundle with an additional structure called an anchor map, whose purpose is to make the vector bundle~~s~~ behave very much like the tangent bundle. In fact the Lie algebroid is sometimes called an alternative tangent space, and this is due to the presence of the anchor map.

Let us set up a **coordinate system** $\{x^\mu\}$ ~~on~~ for a chart ~~in~~ on a manifold $\mathcal{M}$. Let $\{e_a\}$ be a **basis of sections** (vector fields) ~~in~~ a this chart~~,~~ of of the Lie algebroid $(E, \rho, [\cdot,\cdot]_E)$. In this chart we can characterize the anchor map and Lie bracket by **structure functions** as

$$[e_a(x), e_b(x)]_E = T_{ab}{}^c(x) e_c(x) \tag{1}$$

$$\rho(e_a) = h_a{}^\mu(x)\partial_\mu. \tag{2}$$

The Leibniz identity and the Jacobi identity can be written as differential equations constraining the structure functions. I have written them down in the margin of Weinstein's paper.

3 gauge invariance and relational space-time

In this section we ~~put forth~~ propose a picture of gauge theory ~~a method of conceiving of the standard approach to gauge theory~~ that is superior to the conventional one in that ~~it adheres to the relational approach to space-time as far as possible~~. In fact ~~this approach~~ space-time is identified with the web of relationships manifested when the object ψ is acted upon by the various elements of the 'displacement algebra' denoted $\mathfrak{B}$ in all possible ways.

network

space-time is defined as ~~a relati~~ network of relationships between wave-functions.

The displacement algebra, $\mathfrak{B}$, is a formal free algebra generated by elements denoted $\mathcal{P}_a$ subject to precisely one relation r

$$[\mathcal{P}_a, \mathcal{P}_b] = T_{ab}{}^c \mathcal{P}_c. \tag{3}$$

Thus

$$\mathfrak{B} := \mathfrak{B}_{free}/r. \tag{4}$$

The web of relationships arrived at depends critically on the character of r. which is the grav. field. In fact if $\mathfrak{B}$ is non-Abelian the resulting manifold is not flat; it must have non-vanishing curvature or torsion. The fundamental physical principle adhered to in the upcoming considerations is that **parallelism is an observable**. It follows that the relation $r_{||}^{A}$, to be defined below, is gauge invariant. ~~Meaning~~ this means that if two vectors are parallel as measured by a gauge field A, then they must be parallel as measured by any other gauge field A', related to A by a gauge transformation.

We introduce the gauge algebra, denoted $\mathfrak{A}$, which determines what is *physically real.* Indeed the condition for some event in space-time to be physically real is for it to be gauge invariant. **Observables are gauge invariant, and gauge invariant functions are observables.**

In order to clarify the shift from Absolute to Relational space-time we reproduce the argument from the Absolute space-time point of view first and then translate it later to the Relational point of view.

3.1 Absolute

Assume that space-time is homeomorphic to $\mathbb{R}^4$ with a wave function ψ defined upon it along with a connexion 1-form A introduced with the purpose of defining a relation $(r_{||}^{A})$, called parallelism. We say that two vectors (sometimes called phases) at different points on $\mathbb{R}^4$ are parallel with respect to the connexion A by writing

$$\psi|_P \;\; r_{||}^{A} \;\; \psi|_Q. \tag{5}$$

For two neighboring points we can parallel transport a vector $\psi|_P$ as follows

$$\psi|_P \xrightarrow{1+igA_P} \psi|_Q. \tag{6}$$

the bundle for which $\mathbb{R}^4$ is the base. => Need to introduce more notations?

each fibre at each pt of $\mathbb{R}^4$ by same ammount

Now we actively *gauge rotate the whole of* $\mathbb{R}^4$ providing each point with a new gauge convention

$$\psi|_P \xrightarrow{S_P} \psi'_P \tag{7}$$
$$\hat{\psi}|_Q \xrightarrow{S_Q} \hat{\psi}'|_Q \tag{8}$$

subject perhaps only to some mild continuity or differentiability requirements. If parallelism as just defined is to have physical legitimacy as a relation, it must satisfy the condition that there exist a relation, $r_{||}^{A'}$, such that the gauge rotated vectors are also parallel, i.e. that

$$\psi'_P \;\; r_{||}^{A'} \;\; \hat{\psi}'|_Q. \tag{9}$$

It is a simple task to find A' by construction. From the relations [1]

$$\hat{\psi}'|_Q = (1 + igA')\, \psi'|_P \tag{10}$$
$$\psi'|_P = S_P\, \psi|_P \tag{11}$$
$$\hat{\psi}'|_Q = S_Q(1 + igA)\psi|_P, \tag{12}$$
$$S_P = 1 + ig\Lambda|_P, \tag{13}$$

[1] The gauge field is always evaluated at P.

· Add a simple physical section first showing how a consistent view of matter described by QM ⇒ gauge theory. E.g. use Ref. Felsaeger.

Take the reader through the various generalizations to gauge theory proposed by the Chinese authors.

we deduce the fundamental relationship

$$S|_Q(1+igA)S^{-1}|_P = (1+igA'). \tag{14}$$

Calculating the left hand side yields

$$\begin{aligned} &(1+ig\Lambda|_Q)(1+igA)(1-ig\Lambda|_P) && (15)\\ &= [1+igA+ig\Lambda|_Q+(ig\Lambda|_Q)(igA)](1-ig\Lambda|_P) && (16)\\ &= 1+igA+ig\Lambda|_Q+(ig\Lambda|_Q)(igA)-ig\Lambda|_P+ && (17)\\ &(igA)(-ig\Lambda|_P)+(ig\Lambda|_Q)(-ig\Lambda|_P)+(ig\Lambda|_Q)(igA)(-ig\Lambda|_P). \end{aligned}$$

Insert the commutative diagram for the relations.

The last two terms are $O(\Lambda^2)$, the third and fifth terms combine to provide the expected $\partial\Lambda$ term, the first term cancels with a correspondent on the other side. The fourth and sixth terms give the commutator term and a 2-form term that we throw away.

$$igA' = igA + ig\partial_\mu \Lambda dx^\mu - g^2[\Lambda, A] \tag{18}$$

or

def. sign

$$\delta A := A' - A = d\Lambda - ig[\Lambda, A], \tag{19}$$

In (18) we have constructed the A' required by the condition that $r^A_{||}$ be a physical relation and in (19) we have rewritten it as a transformation that provides the proper change of the gauge potential when going from one gauge convention to another. [2]) This is called ~~the~~ a gauge transformation of the gauge potential. It can further be rewritten in terms of the gauge covariant exterior derivative $D = d - igA$.

$$\delta A = D\Lambda \tag{20}$$

3.2 Relational space-time and a view to quantum theoretic space-time

→ 2 poss. for relational i) Emergent ii) non-emergent.

Now that we have reviewed the calculation from the absolute space-time point of view, we abstract the procedure to adhere to the relational point of view, and the above diagram interpreted in space-time is translated to an abstract commutative mathematical diagram. The change is a very significant one. Instead of thinking of a nice classical space $\mathbb{R}^4$ as "out there;" a place upon which fields live, once considerers the gauge theory as living upon the translation group of $\mathbb{R}^4$, denoted $\mathfrak{B}(\mathbb{R}^4)$. This translations group is homeomorphic to $\mathbb{R}^4$ itself, so it would seem that we have made ~~little difference~~ no progress, but this is deceptive. In fact, now we can

i Think of the manifold as a set of relations between wave-functions. A manifestly relational meaning for space-time.

What about Mach? Can his vision be restored.

[2]In the course of this computation, several $O(dx^2)$ (or equivalently two-forms) appear that are ignored, see page 59 of volume 22 of my notes. It might be interesting to consider their meaning.

ii Think of a point of space as being *manifested* by the action of one of the displacement algebra elements, instead of something existing a priori. This latter is imperative if we are to have a traditional quantum mechanical understanding of the underlying space-time.

iii Allow for the possibility of replacing the commutative displacement algebra of translations of $\mathbb{R}^4$ to any algebra we wish, call it $\mathfrak{B}$, in the spirit of Yang-Mills theory.

iv Think of the possibility, made possible by the 'manifested' nature of the space-time, of the dimension of the physically (kinematically) accessible Hilbert space evolving. Banks and others have explored some of this terrain, but it is largely a mystery. Furthermore, if relations define space-time, it is not possible for space-time itself to cut them off. Everything is related to everything else and so causal disconnection is only an approximation. Things may become exponentially approximately disconnected, but not altogether. [3]

emph →

← this could cause serious problems

First of all, the algebra $\mathfrak{B}$ is a Lie algebroid generated by the operators $\{\mathcal{P}_a\}|_{a=1}^{N}$, where N is the dimension of space-time. We restrict ourselves to the Lie algebra case $[\mathcal{P}_a, \mathcal{P}_b] = T_{ab}{}^{c}\mathcal{P}_c$. The wave-function ψ is an element of the Hilbert space $\mathcal{H}$. The gauge field and the gauge parameter are both elements of the gauge algebroid $\mathfrak{A}$.

The operator that generates a gauge transformation at the point $x' = (1 + \epsilon^a\mathcal{P}_a)x$ is

$$(1 + \epsilon^a\mathcal{P}_a)(1 + ig\Lambda)(1 - \epsilon^b\mathcal{P}_b) \tag{21}$$

From this we can write the analogue of equation (10) and following

$$\begin{aligned}
\hat{\psi}' &= (1 + igA')(1 + ig\Lambda) && (22)\\
\psi' &= (1 + ig\Lambda)\psi && (23)\\
\hat{\psi}' &= (1 + \mathcal{P})(1 + ig\Lambda)(1 - \mathcal{P})(1 + igA)\psi. && (24)\\
& && (25)
\end{aligned}$$

After a computation similar to the one following (10), we find that,

$$\delta A = [\mathcal{P}, \Lambda] + (ig)[\Lambda, A] \tag{26}$$

or

$$\delta A = [\mathcal{P}^T, \Lambda] \qquad \mathcal{P}^T = \mathcal{P} - igA. \tag{27}$$

Now we calculate the field strength. We have to stay alert to the fact that torsion has been introduced into the base manifold thought the non-Abelian algebra $\mathfrak{B}$, and that this torsion does not allow the existence of parallelograms at arbitrary locations on the manifold; thus an extra term, second order in smallness, must be added to the usual computation. ← through

[3] If this is true there ~~is~~ may be no horizon problem and no need for inflation.

Here we put a figure of a cracked parallelogram.

More specifically, we analytically attempt to make a complete circuit (parallelogram) using the ingredients of the form $(1+\epsilon\mathcal{P}_a)$. We calculate the product of the two expressions

$$(1+\epsilon'\mathcal{P}_a)(1+\epsilon\mathcal{P}_b) \tag{28}$$

$$(1-\epsilon'\mathcal{P}_a)(1-\epsilon\mathcal{P}_b) \tag{29}$$

and calculate the result to order ϵ^2, getting the following terms

$$\begin{aligned} 1 &+ \epsilon'\mathcal{P}_a + \epsilon\mathcal{P}_b + \epsilon'\epsilon\mathcal{P}_a\mathcal{P}_b - \epsilon'\mathcal{P}_a - \epsilon'^2\mathcal{P}_a^2 \\ &- \epsilon'\epsilon\mathcal{P}_a\mathcal{P}_b - \epsilon\mathcal{P}_b - \epsilon'\epsilon\mathcal{P}_b\mathcal{P}_a - \epsilon^2\mathcal{P}_b^2 + \epsilon\epsilon'\mathcal{P}_a\mathcal{P}_b + O(\epsilon^3). \end{aligned} \tag{30}$$

Now we assume the relation

$$\mathcal{P}_a^2 = 0 = \mathcal{P}_b^2, \qquad \textit{no sum} \tag{31}$$

from which we get

$$1+\epsilon\epsilon'[\mathcal{P}_a,\mathcal{P}_b] =: 1+\epsilon\epsilon' T_{ab}{}^{c}\mathcal{P}_c, \tag{32}$$

which is the expression for the tiny displacement to close the small parallelogram. The $T_{ab}{}^{c}$ is the torsion tensor and, of course varies from point to point on the manifold.

Now we can attempt to calculate the field strength by making a complete circuit and finding the change in phase at one point, before and after a parallel traversal. However instead of chasing vectors around circuits we can make the following equivalent calculation.

$$[\mathcal{P}_a^T,\mathcal{P}_b^T] =: W_{ab}^{\mathfrak{I}} = W_{ab}^{\mathfrak{I}\ c}\,\mathcal{P}_c^T \tag{33}$$

Where W is the field strength for the total algebra $\mathfrak{I}$. The right hand side can further be written

$$W_{ab}^{\mathfrak{I}\ c}\,\mathcal{P}_c^T = W_{ab}^{\mathfrak{I}\ c}\,\mathcal{P}_c - igW_{ab}^{\mathfrak{I}\ c}\,A_c. \tag{34}$$

Direct computation of the left hand side gives

$$[\mathcal{P}_a^T,\mathcal{P}_b^T] = [\mathcal{P}_a,\mathcal{P}_b] - ig[\mathcal{P}_a,A_b] - ig[A_a,\mathcal{P}_b] - g^2[A_a,A_b]. \tag{35}$$

Combining these two and using the additivity of the field strengths,

$$W_{ab}^{\mathfrak{I}\ c}\,\mathcal{P}_c^T = T_{ab}{}^{c}\mathcal{P}_c^T - igF_{ab}{}^{c}\mathcal{P}_c^T, \tag{36}$$

we finally conclude that

$$F_{ab} = [\mathcal{P}_a,A_b] - [\mathcal{P}_b,A_a] - ig[A_a,A_b] - igT_{ab}{}^{c}A_c, \tag{37}$$

which is the standard relation between the Yang-Mills field strength and the gauge field in the presence of torsion.

4 Birkhoff-Santilli mechanics and groupoids

Here we enrich some old ideas of R. M. Santilli with the modern notions of algebroid symmetry in order to produce a coherent generalized gauge theory of particles moving in a gravitational field.

4.1 Birkhoff-Santilli Mechanics

Here we review some of the basic facts about Santilli's generalization of Hamiltonian mechanics based on a comment of Birkhoff. We follow the excellent monograph [?]. It appears that the particular classification used by Santilli are quite consistent with the form that generalized gauge theory of gravity has taken so far. Santilli argues that the Lorentz force is an extremely general self-adjoint force with a potential. Due to the classification of the teleparallel equation as non-potential, due to its quadratic dependence on the velocity, it is implied that the correct formalism for gauge gravity is the Birkhoff-Santilli one as opposed to the Hamiltonian one.

The teleparallel equation of motion for particles in an external gravitational field is

$$\dot{\mathcal{P}}_b = \left(\tfrac{1}{m} F^a{}_{bc} \eta^{cd}\right) \mathcal{P}_a \mathcal{P}_d. \tag{38}$$

By inspection we can see that the force is quadratic in the velocity and therefore cannot satisfy the conditions of self adjointness. In particular the force is a non-potential one and should therefore be treated via the techniques of Birkhoff-Santilli mechanics.

The Birkhoff equation of motion is

$$\Omega_{ab}(t,a)\dot{a}^b - \left[\frac{\partial B(t,a)}{\partial a^a} + \frac{\partial R_a(t,a)}{\partial t}\right] = 0, \tag{39}$$

where

$$\Omega_{ab}(t,a) = \frac{\partial R_b(t,a)}{\partial a^a} - \frac{\partial R_a(t,a)}{\partial a^b}. \tag{40}$$

Now some remarks

-
-
-

4.2 algebroid symmetry

Here we determine what it means for a dynamical system to have an algebroid symmetry. Ultimately, this must be done via the variational principle. First however, we will study symmetries on the level of the equations of motion.

5 Grassmann variables

After a singular search for a Lagrangian description of the Wong equations for Yang-Mills theory, which culminated in questioning the applicability of the standard variational approach, an extremely elegant solution was found in [?]. While the authors are concerned with supersymmetry throughout the paper, the section explicitly dealing with the Wong equations has nothing particularly to do with supersymmetry. While the symmetry is not present, the Lagrangian *does* contain Grassmann variables that play the role of the 'square root' of the gauge currents.

6 What is an observer?

The nature of an observer is a concept that is quite confused at the present moment in science. We review the nature of the concept of observer in two different theories: quantum theory and general relativity. We begin with quantum theory. In this schema the concept of observer has reached its most precise formulation.

Simply, an observer in quantum mechanics is a system capable of affecting a quantum measurement. At this point we have to decide if the observer will include the environment.

$$\begin{array}{ccc} A & \stackrel{a}{\to} & B \\ \downarrow b & & \downarrow c \\ C & \stackrel{d}{\to} & D \end{array}$$

7 BRST cohomology

Let $\mathfrak{g}$ be a Lie algebra with structure constants $f^{ab}_{\ \ c}$ with respect to the basis $\{T^a\}$. Now we introduce **anti-ghosts** β^a, and **ghosts** γ_a, which transform in the adjoint and co-adjoint representations of $\mathfrak{g}$ respectively. It is assumed that the ghosts and anti-ghosts satisfy

$$\{\gamma_a, \beta^b\} = \delta_a^{\ b} \quad \{\gamma_a, \gamma_b\} = 0 = \{\beta^a, \beta^b\} \tag{41}$$