Noncommutativity in Rindler space

G.S. Karatheodoris

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1 Introduction

From one point of view noncommutative field theories arise as an attempt to include one feature of quantum gravity, the absence of well defined localized points due to quantum fluctuations, in a concrete setting. This feature is expected to be present in any quantum theory of gravity due to the old argument that black hole creation will intervene in any attempt to operationally define the location of spacetime points beyond an accuracy $\Delta x \sim l_P \sim 10^{-33}cm$. For a nice review of this argument see (cite bahns, doplicher et. al.) section 1. The reader of this argument will notice that it applies equally well to space and time coordinates. In practice however, the construction of consistent space-space noncommutative theories is much easier to implement than those of the space-time variety. In fact, whether the latter theories are quantum mechanically consistent is a subject of current debate in the literature–a debate we do not wish to enter, as we have not carefully studied the proposal in [?] However, form the point of view of string theory, the lack of unitarity in spacetime-noncommutative field theories is given a particularly simple interpretation in terms of the absence of a decoupling limit which only leaves massless string states. The lack of unitarity is then interpreted as the obvious manifestation of the fact that important dynamical states (the massive string states that did not decouple) are being ignored in a pure field theory limit.

In this work we will attempt to define the noncommutative Rindler wedge

2 Classical Rindler space

Consider $(\mathbf{R}^2, \eta) =: \mathcal{M}$ with coordinates $\{x, t\}$. Classical Rindler space is defined by the coordinate transformation

$$\begin{aligned} x &= a^{-1} e^{a\xi} cosh(a\eta) \\ t &= a^{-1} e^{a\xi} sinh(a\eta) \end{aligned} \tag{1}$$

Which brings the Minkowski metric into the conformally related form

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) \tag{2}$$

The geometry is displayed by the following spacetime diagram

This geometry has the same isometry algebra as $\mathcal{M}$, namely the Poincaré group in two dimensions, $\mathcal{P}(2)$. It is generated by the Killing vectors

$$\begin{aligned} V_1 &= e^{a(\xi+\eta)}(\partial_\eta + \partial_\xi) \\ V_2 &= e^{a(\xi-\eta)}(\partial_\eta - \partial_\xi) \\ V_3 &= \partial_\eta \end{aligned} \tag{3}$$

satisfying

$$[V_1, V_3] = V_3 \qquad [V_2, V_3] = -V_2 \qquad [V_1, V_2] = 0 \tag{4}$$

The timelike killing vector is obviously V_3 in the right wedge **R** and $-V_3$ in the left wedge **L**. The physics in **L** is the time reversal of the physics in **R** as is well known. Note that the Killing vectors are defined on **R**, or with appropriate sign changes on **L**, and are not globally defined on $\mathcal{M}$.

2.1 The Unruh effect

There are at least two physically distinct derivations of the Unruh effect. The most widely known relies on the existence of both Rindler wedges, **L** and **R**. We will sketch the argument. Consider free scalar field theory on $\mathcal{M}$, with equation of motion

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\phi = 0$$

Solutions can be decomposed into a basis consisting of positive frequency eigenfunctions of the timelike Killing vector $\mathcal{L}_{\partial_t}$.

$$u_k = (4\pi\omega)^{-\frac{1}{2}} e^{i(kx-\omega t)}$$

The field is expanded as

$$\phi(t,x) = \sum_k [a_k u_k(t,x) + a_k^\dagger u_k^*(t,x)]$$

and the Minkowski vacuum defined by

$$a_k|0_M> = 0 \qquad \forall k$$

The Fock space is generated by $a_k^\dagger$ acting in all ways on $|0_M>$.

Alternatively one can write a basis of eigenfunctions of positive frequency with respect to $\mathcal{L}_{\pm\partial_\eta}$, where $+\partial_\eta$ is timelike in **R** and $-\partial_\eta$ timelike in **L**. These modes are

$${}^{\mathbf{R}}u_k =$$

3 Noncommutative geometry

We present here a brief review of the formalism of noncommutative geometry in order to introduce some notation and select results. As fermions will not be introduced in this study we will have no need of a Dirac operator and will confine ourselves to a formalism in which there are three main elements: algebra $\mathcal{A}$, Derivations on the algebra, and a definition of integration.

The **algebra** $\mathcal{A}$ will be required to be associative with an antiholomorphic involution. As per usual it will be presented in the form of a free associative algebra (think of all formal polynomials) generated by given hermitian elements a^i such that $(a^i)^\dagger = a^i$ subject to certain relations, given in the form of commutators $[a^i, a^j] = i\xi^{ij}$ with $\xi^{ij} \in \mathbf{R}$.

$$\mathcal{A} = \mathcal{A}_{Free}/([a^i, a^j] - i\xi^{ij} = 0)$$

The **derivations**, ∂_i, on $\mathcal{A}$ are required to satisfy

$$\partial_i(ab) = (\partial_i a)b + a(\partial_i b) \quad \forall a, b \in \mathcal{A} \tag{5}$$

and

$$\int Tr\partial_i a = 0$$
$$\int Tr[a, b] = 0 \tag{6}$$

for elements a, b with "appropriate falloff" at all boundaries and infinity. These last two conditions will be violated for the fields we will be considering in Rindler space. Derivations are inner if they can be written as a commutator and outer otherwise. **Integration**, the notation for which appears above, is required to arise always in conjunction with the trace.

4 The Noncommutative Rindler algebra

We start by assuming that the variables x and t satisfy the relation

$$[x, t] = i\theta_0 \tag{7}$$

where θ_0 is a constant Poisson structure. It is convenient to write a map from Minkowski to exponentiated Rindler coordinates in the following form

$$e^{2a\xi} = a^2(x^2 - t^2) = \frac{a^2}{2}\left((x+t)(x-t) + (x-t)(x+t)\right) \tag{8}$$

$$e^{2a\eta} = \frac{1}{2}((x+t)(x-t)^{-1} + (x-t)^{-1}(x+t)) \tag{9}$$

where this is to be interpreted as an operator equation between *hermitian* operators (this is the reason for the Weyl ordering of the right hand side of the second equation. We find, after a little calculation, that the algebra induced by is

$$[e^{2a\xi}, e^{2a\eta}] = 4ia^2\theta_0 e^{2a\eta}$$

Now using the the definition $\ln(X) = \sum_{n=1}^{\infty}(-1)^n(X)^n$ for the log we deduce the relation

$$[e^{2a\xi}, 2a\eta] = 4ia^2\theta_0$$

which states that the variables $e^{2a\xi}$ and $2a\eta$ are canonically conjugate. The relation

$$[2a\xi, 2a\eta] = i\theta(\xi, \eta) = 4ia^2\theta_0 e^{-2a\xi} \tag{10}$$

then follows immediately. We interpret this as the algebra of the noncommutative Rindler wedge induced by canonical noncommutativity on the Minkowski plane.

One might at this point ask whether this C^*-algebra is amenable to representation by an associative star product. A sufficient condition is $\nabla_i \theta^{jk} = 0$, where ∇_i is the connection associated with some metric. Writing

$$i\theta(\xi,\eta) = i\frac{4a^2\theta_0}{\sqrt{-g}}$$

where g is the determinant of the Minkowski metric in Rindler coordinates we see that the condition is clearly fulfilled. It is a pleasing circumstance that the form of the commutator allows a nice change of variables (cite Fosco) which leads to an exact expression for the star product. We now review this change of variables. In the following we use the notation of (Fosco) in order to keep the equations clean. The relation with our variables is clear and will be explicitly written in the sequel.

Consider the algebra

$$[x^1, x^2]_\star = x^1 \star x^2 + x^2 \star x^1 = i\theta(x^1) = i\theta_0 t(x^1)$$

where $t(x^1) = \frac{1}{\sqrt{-g}}$. Multiplying both sides of the equation by $\frac{1}{\sqrt{t(x^1)}}$ we arrive at

$$y^1 \star y^2 - y^2 \star y^1 = i\theta_0$$

where

$$y^1 = x^1$$

$$y^2 = \frac{1}{\sqrt{t(x^1)}} \star x^2 \star \frac{1}{\sqrt{t(x^1)}}$$

The algebra of the y's can be realized by the Moyal star product. In fact we have

$$f(y) \star g(y) = f(y) * g(y) = exp\{\frac{i}{2}\theta_0 \epsilon^{jk} \frac{\partial}{\partial y^j}\frac{\partial}{\partial \bar{y}}\} f(y) g(\bar{y})|_{y=\bar{y}}$$

The inverse transformation is

$$x^1 = y^1 \tag{11}$$

$$x^2 = \sqrt{t(y^1)} * y^2 * \sqrt{t(y^1)} \tag{12}$$

A dramatic simplification arises due to the fact that we can *directly compute the Moyal products* on the right hand side of the last equation to find

$$x^1 = y^1 \tag{13}$$

$$x^2 = y^2 t(y^1) \tag{14}$$

The noncommutativity has now disappeared in the transformation. With this transformation in hand we can find an explicit star product realization of the algebra (10), define derivations and integration, and write field theories on the Noncommutative Rindler wedge. We write the star product and define derivations and integration below, leaving the field theory construction for the next section.

First we turn to the task of constructing an explicit formula for the C^*-algebra (10). It is easily checked that this algebra is satisfied if we write

$$\begin{aligned} f(x) \star g(x) &:= f(y^1, t(y^1)y^2) * g(y^1, t(y^1)y^2) \\ &= \exp\big(\frac{i}{2}\theta_0 \epsilon^{jk} \frac{\partial}{\partial y^j}\frac{\partial}{\partial \bar{y}^k}\big) f(y^1, t(y^1)y^2) g(\bar{y}^1, t(\bar{y}^1)\bar{y}^2)|_{y=\bar{y}} \end{aligned} \tag{15}$$

Now we simply apply the chain rule to get

$$f(x)\star g(x) = \exp\frac{i\theta_0}{2}\Big[t(\bar{x}^1)\frac{\partial}{\partial x^1}\frac{\partial}{\partial \bar{x}^2} - t(x^1)\frac{\partial}{\partial x^2}\frac{\partial}{\partial \bar{x}^1} + \Big(x^2\frac{t'(x^1)}{t(x^1)}t(\bar{x}^1) - \bar{x}^2\frac{t'(\bar{x}^1)}{t(\bar{x}^1)}t(x^1)\Big)\frac{\partial}{\partial x^2}\frac{\partial}{\partial \bar{x}^2}\Big] \tag{16}$$

Specializing to the case of interest

$$x^1 = 2a\xi \qquad y^1 = x^1 = 2a\xi$$

$$x^2 = 2a\eta \qquad y^2 = e^{x^1}x^2 = e^{2a\xi}2a\eta$$

$$t(x^1) = e^{-2a\xi}$$

we find that

$$f(x)\star g(x) = exp\frac{i\theta_0}{2}\Big[e^{-x_1}\frac{\partial}{\partial x^1}\frac{\partial}{\partial \bar{x}^2} - e^{-x^1}\frac{\partial}{\partial x^2}\frac{\partial}{\partial \bar{x}^1}\big(-x^2e^{-x^1} + \bar{x}^2e^{-x^1}\big)\frac{\partial}{\partial x^2}\frac{\partial}{\partial \bar{x}^2}\Big] \tag{17}$$

This is the star product that realizes the algebra of the Rindler wedge, written in terms of the dimensionless variables $\{x^1, x^2\}$.

Because the noncommutativity has vanished from the map , we can define integration in the usual way using the Lorentz covariant measure

$$d\mu = \sqrt{-g}dx^1dx^2 = e^{x^1}dx^1dx^2 \tag{18}$$

This is also natural from the noncommutative point of view as

$$\int dy^1dy^2\tilde{f}(y^1,y^2) * \tilde{g}(y^1,y^2) = \int dx^1dx^2\frac{1}{t(x^1)}f(x^1,x^2)\star g(x^1,x^2)$$

Inner derivatives are obvious for the y-algebra, they are

$$i\theta_0[y^i, f(y)]_* = \frac{\partial}{\partial y^i}f(y) \tag{19}$$

These can be transformed easily to the dimesionless Rindler variables due again to the commutativity of the transformations (7) (8). The result is

$$\begin{aligned} \partial_{y^1} &= \partial_{x^1} - x^2\partial_{x^2} \\ \partial_{y^2} &= e^{-x^1}\partial_{x^2} \end{aligned} \tag{20}$$

While the direct transformation (8) and (9) are quite simple (and are all we need to define one simple example of noncommutative field theory on the

Rindler wedge), their inverses will be more complicated. We will calculate them now. Defining

$$x_\pm = \frac{1}{\sqrt{2}}(x \pm t) \qquad A = x_+ x_- \quad B = x_+ x_-^{-1}$$

we see that (8) and (9) become

$$\begin{aligned} e^{2a\eta} &= B + B^\dagger \\ \frac{1}{a^2} e^{2a\xi} &= A + A^\dagger \end{aligned} \tag{21}$$

while also

$$\begin{aligned} A - A^\dagger &= -i\theta_0 \\ B - B^\dagger &= i\theta_0 A^{-1} B \end{aligned} \tag{22}$$

so that after some trivial algebra we find

$$\begin{aligned} x_+ &= \sqrt{AB^\dagger} \\ x_- &= \sqrt{B^{-1}A} \end{aligned} \tag{23}$$

where A and B explicitly are

$$\begin{aligned} A &= \left(\frac{-i\theta_0}{2}\right) Q \\ B &= 2\left(1 + Q^{-1}\right)^{-1} u \end{aligned} \tag{24}$$

and

$$Q := (1 + \frac{iv}{a^2\theta_0}) \tag{25}$$

$$v := e^{2a\xi} \tag{26}$$

$$u := e^{2a\eta} \tag{27}$$

5 Another representation of the algebra

The map $\mathcal{S}$ that maps operators to symbols of operators has many equally valid definitions differing by ordering perscriptions. Here we explore one based on normal ordering using the rule "x^1" to the left. The motivation for this is to make clear the consistency of a cutoff prescription that will be introduced later. First note that all elements of $\mathcal{A}_R$ can be written in normal ordered form, as the right hand side of the commutator is only a function of x^1. We define the symbol map $\mathcal{S}$ on an element of $\mathcal{A}_R$ a simply by normal ordering it and replacing the operators in the result by c-number coordinates. Using this prescription a star product can be calculated as

$$A_N \star B_N \equiv \mathcal{S}[\mathcal{S}^{-1}(A_N)\mathcal{S}^{-1}(B_N)]$$

The subscript N represents a normal ordered element. We can also introduce the matrix elements of A: $A(x^1,x^2) \equiv < x^1|A(x^1,x^2)|x^2 >$, noticing that

$$A(x^2,x^2) = < x^1|x^2 > A_N(x^1,x^2). \tag{28}$$

The basic definition of $\star$ is given by

$$(A \star B)(x^1,x^2) \equiv \int d\tilde{x}^1 d\tilde{x}^2 A(x^1,\tilde{x}^2)B(\tilde{x}^1,x^2) \tag{29}$$

using (28) we find that

$$(A \star B)(x^1,x^2) = \tag{30}$$

$$\int \frac{dx^1 dx^2}{8\pi a^2\theta_0} e^{x^1} exp\{\frac{-i}{4a^2\theta_0}(e^{\tilde{x}^1} - e^{x^1})(\tilde{x}^2 - x^2)\} A(x^1,\tilde{x}^2)B(\tilde{x}^1,x^2) \tag{31}$$

or with an obvious change of variables to

$$(A \star B)(q) \equiv (8\pi a^2\theta_0)^{-1} \int d^2q \quad e^{\frac{-i}{4a^2\theta_0}\Delta\tilde{q}^1\Delta\tilde{q}^2} A(q^1,\tilde{q}^2)B(\tilde{q}^1,q^2) \tag{32}$$

where $\Delta\tilde{q}^i \equiv \tilde{q}^i - q^i$. The derivatives are then

$$D_1 := \tag{33}$$

5.1 The near horizon behavior

The Rindler coordinates become ill defined at the horizon and so the determinant of the covariant components of the metric goes to zero. ¿From the form of the noncommutativity parameter in the Rindler frame we see that this implies extremely intense noncommutativity near the horizon. The function $\theta(x^1) = (4a^2\theta_0)e^{x^1}$ is exponentially divergent at the horizon $x^1 \to -\infty$ and we must understand the nature of this pathology and its resolution.

If there were a maximum allowed acceleration in nature we could justify cutting off the integral at a very large but finite value of x^1. In fact, such a maximal acceleration has been argued to exist for quite some time. The statement of the mechanism is quite simple in the context of the Unruh effect. As the acceleration is increased the accelerating object absorbs Unruh radiation at higher temperature. As seen by the inertial observer this corresponds to the accelerating object emitting quanta at said temperature and these quanta are emitted at a cost of energy to the accelerating mechanism.

We now adopt this physically motivated acceleration cutoff on the level of kinematics, i.e. with an ad hoc cutoff. This step is not as trivial as it might seem since the noncommutative algebra must close within the truncated region. We now wish to show this. The geometry is (insert diagram)

We call the truncated region R^ϵ and the putative algebra $\mathcal{A}_{R^\epsilon}$. The modification to the star product is clearest in the integral kernel representation, it amounts to replacing the region of integration R with R^ϵ. Also, the normal ordered symbols of operators A are replaced by $H(x^1 + \frac{1}{\epsilon})A$. Closure is now obvious.

It is quite a popular procedure to truncate noncommutative algebras algebra

5.2 Bases

There are a variety of bases one might consider picking for R^ϵ: plane wave, Gaussian, delta function. In order to gain some intuition for the algebra A^ϵ we can explore these bases. When dynamics is introduced later on we will of course be interested in convenient bases of solutions to the classical field equations.

5.2.1 Plane waves

Something amusing happens when we attempt to multiply two plane waves in Rindler coordinates. Such waves take the form

$$e^{ik\cdot x} = e^{ik_1 x^1} e^{ik_2 x^2} = (q^1)^{ik_1} e^{ik_2 q^2}$$

and the product is

$$\begin{aligned}(e^{ik\cdot y} \star e^{ik'\cdot z})(q^1, q^2) &= (8\pi a^2\theta_0)^{-1} \int d\tilde{q}^1 d\tilde{q}^2 e^{\frac{-i}{4a^2\theta_0}\Delta\tilde{q}^1\Delta\tilde{q}^2} (q^1)^{ik_1} e^{ik_2\tilde{q}^2} (\tilde{q}^1)^{ik'_1} e^{ik'_2 q^2} \\ &= \int_\delta^\infty d\tilde{q}^1 e^{\frac{i}{4a^2\theta_0} q^2\Delta\tilde{q}^1} (q^1)^{ik_1} (\tilde{q}^1)^{ik'_1} e^{ik'_2 q^2} \int_{-\infty}^\infty d\tilde{q}^2 e^{\frac{-i}{4a^2\theta_0}\tilde{q}^2\Delta\tilde{q}^1} e^{ik_2\tilde{q}^2} \\ &= e^{\frac{-i}{4a^2\theta_0} q^1 q^2} (q^1)^{ik_1} e^{ik'_2 q^2} \int_\delta^\infty d\tilde{q}^1 e^{\frac{i}{4a^2\theta_0} q^2\tilde{q}^1} (\tilde{q}^1)^{ik'_1} \int_{-\infty}^\infty d\tilde{q}^2 e^{\frac{-i}{4a^2\theta_0}\tilde{q}^2\Delta\tilde{q}^1} e^{ik_2\tilde{q}^2} \\ &= 8\pi a^2\theta_0 (q^1)^{ik_1} (q^1 + 4a^2\theta_0 k_2)^{ik'_1} e^{i(k'_2+k_2)q^2} \end{aligned} \quad (34)$$

5.2.2 Gaussians

5.3 reality conditions

5.4 The integral of a commutator

In the absence of boundaries or in the presence of peculiar restrictions on the space of operators one can integrate the star commutator to get zero. In operator language this corresponds to the trace of a commutator being zero (with appropriate restrictions). In the present context this issue must be addressed if we wish to calculate equations of motion from an action principle. At what expense can we commute the order of functions in and integral? The price will be a boundary term, in analogy with total derivatives, which we now calculate.

We would like to write $[(q^1)^M, (q^2)^N]_\star$ as a total derivative. Using the integral representation we calculate $(q^2)^N \star (q^1)^M$

$$(8\pi a^2\theta_0)^{-1} \int_{R^\epsilon} d\tilde{q}^1 d\tilde{q}^2 e^{\frac{-i}{4a^2\theta_0}\Delta\tilde{q}^1\Delta\tilde{q}^2} (\tilde{q}^2)^N (\tilde{q}^1)^M$$

$$= (8\pi a^2\theta_0)^{-1}\int_\delta^\infty d\tilde{q}^1 e^{\frac{i}{4a^2\theta_0}q^2\Delta\tilde{q}^1}(\tilde{q}^1)^M\int_{-\infty}^\infty d\tilde{q}^2 e^{\frac{-i}{4a^2\theta_0}\tilde{q}^2\Delta\tilde{q}^1}(\tilde{q}^2)^N$$

$$= (8\pi a^2\theta_0)^{-1}\int_\delta^\infty d\tilde{q}^1 e^{\frac{i}{4a^2\theta_0}q^2\Delta\tilde{q}^1}(\tilde{q}^1)^M 2\pi i^N\delta^{(N)}(\frac{\Delta\tilde{q}^1}{4a^2\theta_0})$$

$$= \frac{2\pi i^N}{8\pi a^2\theta_0}(4a^2\theta_0)^{N+1}e^{\frac{-i}{4a^2\theta_0}q^1q^2}\frac{d^N}{d\tilde{q}^{1N}}\left\{e^{\frac{i}{4a^2\theta_0}\tilde{q}^1q^2}(\tilde{q}^1)^M\right\}\Big|_{q^1=\tilde{q}^1}$$

$$= \frac{2\pi i^N}{8\pi a^2\theta_0}(4a^2\theta_0)^{N+1}(q^1)^M(\frac{iq^1}{4a^2\theta_0})^N\sum_{k=0}^N\binom{N}{k}\frac{M!}{(M-k)!}\Big(\frac{iq^1q^2}{4a^2\theta_0}\Big)^{-k}$$

$$(q^1)^M(q^2)^N\sum_{k=0}^N\binom{N}{k}\binom{M}{k}k!\Big(\frac{iq^1q^2}{4a^2\theta_0}\Big)^{-k} \quad (35)$$

Where we have used the relation $\int_{-\infty}^\infty e^{-i\omega t}t^n = 2\pi i^n\delta^{(n)}(\omega)$. The generalization to $(q^2)^N \star f(q^1)$ is

$$(q^2)^N \star f(q^1) = (q^2)^N\sum_{k=0}^N\binom{N}{k}(\frac{iq^2}{4a^2\theta_0})^{-k}f^{(k)}(q^1) \quad (36)$$

When $g(q^2)$ can be written as a power series in q^2 we have

$$g(q^2)\star f(q^1) = \sum_{N=0}^\infty C_N(q^2)^N \star f(q^1) \quad (37)$$

$$= \sum_{N=0}^\infty\sum_{k=0}^N\binom{N}{k}C_N(q^2)^N(\frac{iq^2}{4a^2\theta_0})^{-k}f^{(k)}(q^1) \quad (38)$$

Using the fact that $f(q^1)\star g(q^2) = f(q^1)g(q^2)$ and noticing that this is what appears as the $k=0$ term in the sums we find the following formulas for the star commutators which are indeed total derivatives

$$[(q^1)^M,(q^2)^N]_\star = \frac{d}{dq^1}\Big\{-(q^1)^{M+1}(q^2)^N\sum_{k=1}^N\binom{N}{k}\binom{M}{k}\frac{k!}{(M-k+1)}\Big(\frac{iq^1q^2}{4a^2\theta_0}\Big)^{-k}\Big\} \quad (39)$$

$$[f(q^1),(q^2)^N] = \frac{d}{dq^1}\Big\{-(q^2)^N\sum_{k=1}^N\binom{N}{k}(\frac{iq^2}{4a^2\theta_0})^{-k}f^{(k-1)}(q^1)\Big\} \quad (40)$$

$$[f(q^1),g(q^2)] = \frac{d}{dq^1}\Big\{\sum_{N=0}^\infty\sum_{k=0}^N\binom{N}{k}C_N(q^2)^N(\frac{iq^2}{4a^2\theta_0})^{-k}f^{(k-1)}(q^1)\Big\} \quad (41)$$

Another way to think about this is as follows. We have reviewed that nontrivial commutation relation can be reduced to the canonical case through a change of variables. In the Rindler case the canonical algebra is,

$$[q^1,q^2] = 4a^2\theta_0$$

and the star product in this case is of course the Moyal one,

$$e^{2ia^2\theta_0(\partial_1\tilde{\partial}_2-\partial_2\tilde{\partial}_1)}$$

Thus the boundary term can be computed easily from this differential form, yeilding

$$J^\mu = \sum_{n=0}^{\infty} J_n^\mu \tag{42}$$

$$J_n^\mu = \frac{1}{n!}(\frac{i}{2})^n \theta^{\mu\nu_1}\cdots\theta^{\mu_n\nu_n}\phi_{,\mu_2\cdots\mu_n}\phi_{,\nu_1\cdots\nu_n} \tag{43}$$

A boundary term must now be constructed from this expression in order for the action to satisfy a variational principle. This term is

$$S_b[\phi] = \int_{\partial\mathcal{R}^\epsilon} J^\mu n_\mu dq^2 = \int_{\partial\mathcal{R}^\epsilon} J^1 dq^2 \tag{44}$$

Evaluated on the boundary means $q^1 = \delta$.

$$J_n^1 = \frac{1}{n!}(2ia^2\theta_0)^n\epsilon^{12}\cdots\epsilon^{\mu_n\nu_n}\phi_{,\mu_2\cdots\mu_n}\phi_{,2\nu_2\cdots\nu_n}$$

$$\Rightarrow J^1 = \psi' \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\frac{1}{n!}(2ia^2\theta_0)^n(-1)^i\binom{n-1}{i}(\vartheta^{(n-i)})^2(\psi^{(i)})^2 \tag{45}$$

but, at the monomial level,

$$\phi(q^1,q^2) = \vartheta(q^1)\psi(q^2) \Rightarrow \phi_{,\mu} = \delta_{1\mu}\partial_\mu\vartheta\psi + \delta_{2\mu}\vartheta\partial_\mu\psi \tag{46}$$

6 spacetime uncertainty relations

7 Noncommutative Field Theory on the Rindler wedge

In this section we define field theory on the noncommutative Rindler wedge. Before we begin we note that the issue of the unitarity of space-time noncommutative theories is a subject of continued interest in the literature. We will address in detail the point of view taken toward this issue in the present study in section ??. For the present we will continue formally. We start with the action

$$S[\tilde{\phi}] = \int dy^1dy^2\Big(-\frac{1}{2}(\frac{\partial\tilde{\phi}}{\partial y^2})*(\frac{\partial\tilde{\phi}}{\partial y^2})+\frac{1}{2}(\frac{\partial\tilde{\phi}}{\partial y^1})*(\frac{\partial\tilde{\phi}}{\partial y^1})+\frac{m^2}{2}\tilde{\phi}*\tilde{\phi}+V_*(\tilde{\phi})\Big) \tag{47}$$

in y-coordinates which we interpret as noncommutative scalar field theory on the Rindler wedge in a set of variables that make it possible to write the star product as the Moyal-Weyl one.

It is important to realize that the restriction of the usual action on the noncommutative plane to the Rindler wedge

$$S[\phi] = \int_R dxdt\big(- \frac{1}{2}\frac{\partial\phi}{\partial t} * \frac{\partial\phi}{\partial t} + \cdots\big)$$

is not a well defined object. This is because the Moyal product does not respect the artificial termination of flows corresponding to the vector fields appearing in its definition. This is easily seen by noting that the generator of translations appears in an exponential, and when such an object acts on a function it translates it along the flow of the associated generator. The function that results from such an action, say $(g \circ f)(x)$ will in general have different support than $f(x)$, i.e. $supp(g \circ f) \nsubseteq supp(f)$. Contrary to the claim in (cite Bal) the solution to this particular problem is not impossible: one simply uses as coordinates for the compact region, flows generated by vector fields that have no flux through any portion of the boundary.

There is also an important physical argument that is much more fundamental. First lets consider ordinary local field theory. In this framework we know that local actions must be appended with boundary terms (appropriate to assumed boundary conditions) in order for actions to be additive, in the sense that given two regions of space Ω_1 and Ω_2 with a common boundary the sum of the actions defined on Ω_1 and Ω_2 separately, is equal to the action defined on $\Omega_1 \cup \Omega_2$. That is

$$\int_{\Omega_1 \cup \Omega_2} \mathcal{L} d\mu = \int_{\Omega_1} \mathcal{L} d\mu + \int_{\Omega_2} \mathcal{L} d\mu \tag{48}$$

The same relation can not immediately be assumed to apply to nonlocal (e.g. noncommutative) theories. As an example of this consider a free noncommutative scalar field theory action defined in a region of the Moyal plane Ω with boundary $\partial\Omega$. The action

$$S[\Omega, \phi, \theta] = \int_{\Omega} dxdt\{-\frac{1}{2}\dot{\phi} * \dot{\phi} + \frac{1}{2}\phi' * \phi'\} \tag{49}$$

varying with respect to ϕ yeilds

$$\delta_\phi S[] = \int_\Omega dxdt\ \delta\phi * \{\ddot{\phi} - \phi''\} + \int_{\partial\Omega} ds\ n_i\{\eta^{ij}\delta\phi * \phi_{,j}\} \tag{50}$$

where n_i is normal to the curve $\partial\Omega$ and s parameterizes the same. We wish to obtain the equations of motion from the condition that

$$\delta\phi|_{\partial\Omega} = 0$$

now, since $\delta\phi$ is constant on $\partial\Omega$ we Recall that $\tilde{\phi}(y) = \phi(x(y))$ and that the y coordinates are good on the whole Rindler wedge except on the horizons. The reader can check this by computing the determinant of the Jacobian of the transformations (7) (8). The integral then extends over all real values of the variables $\{y^i\}$ and we do not encounter the above problem. If we impose the usual boundary conditions at the Rindler horizon, the integral of the quadratic Moyal products can be replaced by the integral of ordinary pointwise products yielding

$$S[\tilde{\phi}] = \int dy^1 dy^2 \Big(-\frac{1}{2}\big(\frac{\partial\tilde{\phi}}{\partial y^2}\big)^2 + \frac{1}{2}\big(\frac{\partial\tilde{\phi}}{\partial y^1}\big)^2 + \frac{m^2}{2}\tilde{\phi}^2 + V_*(\tilde{\phi})\Big) \tag{51}$$

The noncommutativity enters into the free theory via the metric and derivatives, and of course into the interacting theory via $V_*(\tilde{\phi})$.

7.1 A simpler way?

$$S[\phi] = \int_{\mathcal{R}^\epsilon} d\mu(q)\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi \star \partial_\nu\phi \tag{52}$$

$$q^1 \in (\delta, \infty) \quad q^2 \in \mathbf{R} \tag{53}$$

After the variation we get

$$\delta S[\phi] = \int_{\mathcal{R}^\epsilon} d\mu(q)\{-\triangle\phi\} \star \delta\phi + \delta S_b^1[\phi] + \delta S_b^2[\phi] \tag{54}$$

where the last two terms are variations of boundary terms, i.e. are total derivatives. Explicitly,

$$\delta S_b^1[\phi] = \frac{1}{2}\int_{\mathcal{R}^\epsilon} d\mu(q)\partial_1\{\delta\phi \star \partial_1\phi + \partial_1\phi \star \delta\phi\} \tag{55}$$

$$\delta S_b^2[\phi] = \frac{1}{2}\int_{\mathcal{R}^\epsilon} d\mu(q)[\triangle\phi, \delta\phi]_\star \tag{56}$$

8 Relation to 'tHooft's S-matrix approach

9 The maximum tension principle

10 Relation to the matrix model

11 Relation to Liouville theory

12 problems with the approach

- Transverse coordinates must be added, or we are just describing a degenerate case.
- Include a section on noncommutative half space.
-

13 Coordinate transformations of symbols

In noncommutative geometry, convenient procedure is to replace operators satisfying a certain algebra $\mathcal{A}$, by their symbols, and implement algebraic relation by introducing a suitable deformed product. These symbols are then often manipulated as if they represent a geometrical continuum. A common situation arises: one attempts to 'change coordinates' on this 'geometrical continuum' without worrying too much about mathematical niceties. To what extent is this justified?

First lets consider the simple example of the noncommutative plane R^2_θ, with generators $\{\hat{x}^i\}$. First we will consider the case of a change of coordinates that **covers the entire plane** and **respects the symmetries** of R^2_θ. The first condition refers to the effect of the coordinate transformation (CT) on the indices labeling the symbols of the operators generating R^2_θ, while the second condition refers to the effect of the CT on the symmetries of the noncommutative algebra R^2_θ. One of the apparent features of noncommutative spaces as geometric objects is that they preserve the symmetries of their commutative limits. As ingredients for field theories, the situation regarding symmetries is more involved, as there can be a mixture of internal and spacetime symmetries in the presence of noncommutativity.