- Review:

1. $(a+a^+)^N = a^{+N} + a^{+(N-2)} f^N_{N-2}(n) + \cdots + a^{+N-2q} f^N_{N-2q}(n) + \cdots + h.c.$

2. Now calculate $f^N_{N-2q}(n)$.

   - Recurrence relations are:

$$\boxed{\begin{aligned} f^{N+1}_{N-2(q+1)}(n) &= (N-q) + n + f^N_{N-2(q+1)}(n) \quad &&\text{- from } a^+ \\ f^{N+1}_{N-2(q+1)}(n) &= f^N_{N-2(q+1)}(n+1) + n f^N_{N-2q}(n) \quad &&\text{- from } a \end{aligned}}$$

   - $$\left.\boxed{\begin{aligned} & f^N_N(n) = 1 \\ & f^N_{N-2}(n) = nN + \frac{N(N-1)}{2} \end{aligned}}\right\}$$ A couple of solutions.

   - The hermiticity conditions yeild further recurrence relations; Setting the a and a^+ coefficients equal gives

$$(N-q) + n + f^N_{N-2(q+1)}(n) = f^N_{N-2(q+1)}(n+1) + n f^N_{N-2q}(n)$$

Define $\Delta f^N_{N-2(q+1)}(n) = f^N_{N-2(q+1)}(n+1) - f^N_{N-2(q+1)}(n)$

Then,

$$\boxed{\Delta f^N_{N-2(q+1)}(n) = (N-q) + n - n f^N_{N-2q}(n)}$$

A few examples

$q = 0$ $\qquad f^N_{N-2}(n+1) - f^N_{N-2}(n) = (N) \cancel{+ n} \cancel{- n \underbrace{f^N_N(n)}_{1}}$,

$$(\cancel{n}+1)N + \cancel{\frac{N(N-1)}{2}} - \cancel{nN} - \cancel{\frac{N(N-1)}{2}} = N$$ ✓ consistent.

$q = 1$ $\qquad f^N_{N-4}(n+1) - f^N_{N-4}(n) = (N-1) + n - n \underbrace{f^N_{N-2}(n)}_{nN + \frac{N(N-1)}{2}}$

$$f^N_{N-4}(n+1) - f^N_{N-4}(n) = (N-1) + n - n^2 N - n\left(\frac{N(N-1)}{2}\right)$$

$$= (-N)\,n^2 + n\left(-\frac{N(N-1)}{2} + 1\right) + (N-1)$$

$$\boxed{\Delta f^N_{N-4}(n) = (-N)\,n^2 + \left(1 - \frac{N(N-1)}{2}\right) n + (N-1)}$$